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CHARACTERISATION OF FUETER MAPPINGS AND THEIR JACOBIANS

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We continue the study of a class of real analytic mappings (christened Fueter mappings) from open subsets of \mathbb{R}^n into \mathbb{R}^n . This class has connections with spaces of holomorphic functions and functions of quaternionic and octonionic variables. We characterise such mappings by a system of partial differential equations and determine conditions for their K -quasiconformality in the sense of Ahlfors.

We also characterise the Jacobian matrices of these mappings. The Jacobian matrices form a family of subgroups of $GL(n, \mathbb{R})$ (which are mutually isomorphic) parametrised by certain projective spaces.

INTRODUCTION

This paper is a continuation of our study of Fueter's interesting class of mappings. The domains and ranges are open subsets of \mathbb{R}^n ($n \geq 2$), and the maps are obtained by certain transformations of complex analytic functions. The mappings are as follows: Let ϕ be a holomorphic mapping whose domain is an open subset of the upper half plane $U = \{z : \text{Im}(z) > 0\}$. The n -dimensional Fueter transform of ϕ , denoted by $F_n(\phi)$, is obtained by substituting in ϕ the expression $(e_1 x_1 + \dots + e_{n-1} x_{n-1}) / (x_1^2 + \dots + x_{n-1}^2)^{1/2}$ for the imaginary unit i . (Here \mathbb{R}^n has coordinate (x_0, \dots, x_{n-1}) with unit vectors e_0, e_1, \dots, e_{n-1}).

One main reason for interest in the 'Fueter maps' stems from the fact that $F_4(\phi)$ and $F_8(\phi)$ are expressible as power series in a quaternionic or octonionic variable (respectively) when ϕ has formally-real expansion around real centres.

Nag *et al.*⁸ had proved that Fueter diffeomorphisms (and the corresponding quaternionic and octonionic mapping classes) from pseudogroups. The main interest has been in modelling C^∞ manifolds on these pseudogroups. In the same paper⁸ a geometrical interpretation of how Fueter diffeomorphisms arise was discussed via certain "rotations" of complex analytic mappings. Using these principles we had been successful in characterising compact hypercomplex and Fueter manifolds in that paper.

In the present study our main purpose is to make a more algebraic attack on the study of these pseudogroups. First of all we are able to characterise Fueter mappings by a set of algebro-differential conditions.

The problem of whether a given C^∞ manifold can be assigned hypercomplex/Fueter structure is of course intimately related to whether the structure group of the tangent bundle of the manifold can be reduced to the group of Jacobians of hypercomplex/Fueter diffeomorphisms. In this paper we therefore study the Lie groups of Jacobian matrices and their corresponding Lie algebras. The results of Nag *et al.*⁸ might therefore be approachable by pure differential geometric methods using the conclusions of the present study.

Imaeda and Imaeda^{6,7}, have also pursued analytic functions of hypercomplex variables, extending work of Fueter *et al.*. In section 4 we describe the connection between the functions treated here and Imaeda's functions.

1 THE FUETER TRANSFORMATION

The precise definition of $F_n(\phi)$ is :

Let D be a region in U (the standard upper half plane in \mathbb{C}) and ' $\mathbb{R}^n = \mathbb{R}^n - \{x_0 - \text{axis}\}$ ' ($n \geq 2$), we set

$$F_n(D) = \{(x_0, x_1, \dots, x_{n-1}) \in \mathbb{R}^n : (x_0, (x_1^2 + \dots + x_{n-1}^2)^{1/2}) \in D\} \\ \subseteq \mathbb{R}^n. \quad \dots(1)$$

Let $\phi : D \rightarrow \mathbb{C}$ be complex analytic with real and imaginary part decomposition $\phi = \xi + i\eta$, then

$F_n(\phi) : F_n(D) \rightarrow \mathbb{R}^n$ is defined by

$$F_n(\phi)(x_0, \dots, x_{n-1}) = \xi(x_0, y) + \sum_{j=1}^{n-1} \frac{e_j x_j}{y} \eta(x_0, y) \quad \dots(2)$$

where $y = (x_1^2 + \dots + x_{n-1}^2)^{1/2} > 0$.

If the holomorphic map ϕ has real boundary values where the real axis abuts D then a direct application of the reflection principle guarantees that $F_n(\phi)$ can be defined real analytically on the revolved domain $F_n(D)$ together with the corresponding portions of the $x_0 - \text{axis}$.

We can also define a Fueter transform on analytic maps of several complex variables. [For the sake of simplicity of notation we consider the case of 2-complex variables].

Let $\phi \equiv (\phi_1, \phi_2) : D \rightarrow \mathbb{C}^2$ be an analytic map.

$$(D \subseteq U^2 = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Im}(z_j) > 0; j = 1, 2\}.)$$

We define its Fueter transform $F_n^{(2)}(\phi) : F_n^{(2)}(D) \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ by

$$\begin{aligned}
 F_n^{(2)}(\phi) & (x_{0,1} + e_1 x_{1,1} + \dots + e_{n-1} x_{n-1,1}, x_{0,2} + e_1 x_{1,2} \\
 & + \dots + e_{n-1} x_{n-1,2}) \\
 & = \left(\xi_1 + \left(\sum_{j=1}^{n-1} e_j x_{j,1} \right) \frac{\eta_1}{y_1}, \xi_2 + \left(\sum_{j=1}^{n-1} e_j x_{j,2} \right) \frac{\eta_2}{y_2} \right) \dots (3)
 \end{aligned}$$

where $\phi_j = \xi_j + i\eta_j$, $y_j = \left(x_{1,j}^2 + \dots + x_{n-1,j}^2 \right)^{1/2}$, $j = 1, 2$.

$$\begin{aligned}
 F_n^{(2)}(D) & = \{x_{0,1} + e_1 x_{1,1} + \dots + e_{n-1} x_{n-1,1}, \\
 & x_{0,2} + e_1 x_{1,2} + \dots + e_{n-1} x_{n-1,2}\}; \\
 & (x_{0,1} + iy_1, x_{0,2} + iy_2) \in D, y_j \text{ as above} \} \subseteq \mathbb{R}^n \times \mathbb{R}^n. \dots (4)
 \end{aligned}$$

Some simple properties are given below.

$$F_n(a\phi) = a F_n(\phi), a \in \mathbb{R} \dots (5)$$

$$F_n(\phi + \psi) = F_n(\phi) + F_n(\psi) \dots (6)$$

$$F_n(\phi \circ \psi) = F_n(\phi) \circ F_n(\psi) \dots (7)$$

$$F_n(j_2 \circ \phi) = j_2 \circ F_n(\phi). \dots (8)$$

(j_n is the conjugation in \mathbb{R}^n i. e., $j_n(x_0, \dots, x_{n-1}) = (x_0, -x_1, \dots, -x_{n-1})$)

$$F_n(\phi^{-1}) = F_n(\phi)^{-1} \dots (9)$$

(whenever ϕ^{-1} is well defined with domain in U).

$$F_n(\phi, \psi) = F_n(\phi), F_n(\psi), \text{ for } n = 4 \text{ or } 8. \dots (10)$$

Remark : A Möbius (conformal) transformation (in dimensions $n > 2$) may not be a Fueter mapping. Indeed even the translation $V \rightarrow V + b$, b not purely real, already fails to be a Fueter map.

*Definition*¹—Consider a differentiable mapping

$$f: D (\subseteq \mathbb{R}^n) \rightarrow \mathbb{R}^n.$$

Define,

$$Sf = \frac{1}{2} (Df + (Df)^T) - \frac{1}{n} \text{Tr}(Df) I_n$$

where Df is the Jacobian matrix of f and $\text{Tr}(Df) = \text{Trace of } Df$.

$\|Sf\| = (\text{sum of the squares of the entries of } Sf)^{1/2}$. f is said to be K -quasiconformal if $(1/\sqrt{n}) \|Sf\| \leq K$.

Proposition 1.1—Suppose $f = F_n(\phi)$ is a Fueter mapping. f is K -quasiconformal if

$$|\eta_y - \eta/y| \leq 2K \text{ over the domain of } \phi = \xi + i\eta. \dots (11)$$

(The result has already been announced in Nag *et al.*⁸).

PROOF : By direct calculation one obtains

$$\|Sf\|^2 = \left(\eta_y - \frac{\eta}{y}\right)^2 \cdot \frac{(2n^2 - 4n)}{n^2}.$$

$$\therefore (1/\sqrt{n}) \|Sf\| = (2n - 4)^{1/2} \cdot n^{-1} |\eta_y - \eta/y| \leq \frac{1}{2} |\eta_y - \eta/y|.$$

The result follows.

2. CHARACTERIZATION OF FUETER MAPPINGS (ANALYTICALLY)

Theorem 2.1—Let $f \equiv (f_0, \dots, f_{n-1}) = F_n(\phi) : F_n(D) \rightarrow \mathbb{R}^n$ be a Fueter mapping.

Then f satisfies the following relations :

$$\delta_0 f_j = -\delta_j f_0 \quad (j > 0) \left[\delta_p = \frac{\delta}{\delta x_p}, p = 0, \dots, n-1 \right] \quad \dots(12)$$

$$\delta_k f_j = \delta_j f_k \quad (j, k > 0) \quad \dots(13)$$

$$\langle \nabla f_0, \nabla f_j \rangle = 0 \quad (j > 0). \quad \dots(14)$$

$$\text{Supertrace of Jac}(f) (= \delta_0 f_0 - \sum_{j=1}^{n-1} \delta_j f_j)$$

$$= (2 - n) \left(f_1^2 + \dots + f_{n-1}^2 \right)^{1/2} / \left(x_1^2 + \dots + x_{n-1}^2 \right)^{1/2} \\ \text{at } (x_0, \dots, x_{n-1}) \quad \dots(15)$$

(in case $n = 2$, (15) becomes $\delta_0 f_0 - \delta_1 f_1 = 0$ which is the second Cauchy-Riemann relation.)

$$(\delta_0 f_0) x_k = \sum_{j=1}^{n-1} (\delta_k f_j) x_j, \quad k > 0 \quad \dots(16)$$

(in case $n = 2$, (16) also reduces to the Second Cauchy-Riemann relation.)

$$\delta_0 f_0 \text{ is a function of } x_0 \text{ and } \left(x_1^2 + \dots + x_{n-1}^2 \right) = Y \text{ only,} \quad \dots(17)$$

equivalently,

$$\frac{\delta_1 (\delta_0 f_0)}{x_1} = \frac{\delta_2 (\delta_0 f_0)}{x_2} = \dots = \frac{\delta_{n-1} (\delta_0 f_0)}{x_{n-1}} \quad \dots(17a)$$

(for $x_1, \dots, x_{n-1} \neq 0$)

$$\frac{\delta_k f_j}{x_k x_j} = \frac{\delta_p f_q}{x_p x_q}$$

for all $p, q, j, k > 0$, $x_j, x_k, x_p, x_q \neq 0$, and $k \neq j, p \neq q$ (18)

$$\frac{\delta_k f_j}{x_k x_j} \text{ is a function of } x_0 \text{ and } \left(x_1^2 + \dots + x_{n-1}^2 \right) = Y \text{ only,} \quad \dots(19)$$

for $k \neq j, k > 0$.

Equivalently,

$$\frac{\delta_1 \left(\frac{\delta_k f_j}{x_k x_j} \right)}{x_1} = \frac{\delta_2 \left(\frac{\delta_k f_j}{x_k x_j} \right)}{x_2} = \dots = \frac{\delta_{n-1} \left(\frac{\delta_k f_j}{x_k x_j} \right)}{x_{n-1}} \quad \dots(19a)$$

for $x_1, x_2, \dots, x_{n-1} \neq 0$

$$f_p(x_0, \dots, x_{n-1}) = \left(\delta_0 f_0 - \left(x_1^2 + \dots + x_{n-1}^2 \right) \frac{\delta_k f_j}{x_k x_j} \right) x_p \quad \dots(20)$$

for $k, j > 0, k \neq j, x_k \cdot x_j \neq 0, p > 0$.

$$\begin{aligned} & y \delta_0^3 f_0 - y^3 \delta_0^2 \left(\frac{\delta_k f_j}{x_k x_j} \right) + 2y \frac{\delta_k (\delta_0 f_0)}{x_k} + \frac{y^2}{x_p} \delta_p \left(\frac{\delta_q (\delta_0 f_0)}{x_q} \cdot y \right) \\ & - 6y \frac{\delta_k f_j}{x_k x_j} - \frac{6y^3}{x_r} \delta_r \left(\frac{\delta_k f_j}{x_k x_j} \right) - \frac{y^4}{x_s} \delta_s \left[\frac{y}{x_r} \delta_r \left(\frac{\delta_k f_j}{x_k x_j} \right) \right] \\ & = 0 \end{aligned} \quad \dots(21)$$

$k \neq j, k, j, r, p, q, s > 0, x_j, x_p, x_q, x_r, x_s \neq 0$,

$$y = \left(x_1^2 + \dots + x_{n-1}^2 \right)^{1/2}.$$

[(21) is unambiguous because of (17a) and (19a)].

$$y^2 \frac{\delta_p (\delta_0 f_0)}{x_p} - 3y^2 \frac{\delta_k f_j}{x_k x_j} - \frac{y^4}{x_q} \delta_q \left(\frac{\delta_k f_j}{x_k x_j} \right) = 0 \quad \dots(22)$$

$x_j, x_k, x_p, x_q \neq 0, k \neq j, j, k, p, q > 0$.

PROOF : The Cauchy-Riemann relation between ξ and η and Laplace's equation $\Delta \eta = 0$ (which is (21)) implies all the above formulae.

Theorem 2.2— $f \equiv (f_0, \dots, f_{n-1}) : \tilde{D} (\subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n)$ is a Fueter map if and only if it satisfies formulas (12) and (17) — (22).

PROOF : Let $y = \left(x_1^2 + \dots + x_{n-1}^2 \right)^{1/2}$

$$\text{Put } \eta(x_0, y) = y \left(\delta_0 f_0 - y^2 \frac{\delta_k f_j}{x_k x_j} \right)$$

(it is unambiguous because of (17), (18) and (19)) then by (20)

$$f_k = \frac{x_k}{y} \eta, k > 0.$$

The equation (12) says

$$\delta_k f_0 = -\delta_0 f_k = -\frac{x_k}{y} \eta_{x_0}, k > 0.$$

Therefore,

$$\frac{\delta_1 f_0}{x_1} = \dots = \frac{\delta_{n-1} f_0}{x_{n-1}}$$

equivalently f_0 depends on x_0 and y only (which means that on x_0 and y constant loci f_0 takes constant values).

One may therefore unambiguously define

$$\xi(x_0, y) = f_0(x_0 + e_1 x_1 + \dots + e_{n-1} x_{n-1}).$$

Then by eqn. (12) $\xi_y = -\eta_{x_0}$

and by eqn. (22) $\xi_{x_0} = \delta_0 f_0 = \eta_y$.

Equation (21) says η is harmonic in the relavent domain of the $x_0 - y$ plane.

Now one verifies that $f = F_n (\xi + i\eta)$ on the relavent domain.

3. JACOBIANS OF FUETER MAPPINGS

Definition—For any $\tilde{k} = (k_1, \dots, k_{n-1}) \in S^{n-2}$, we consider a subgroup of $GL(n, \mathbb{R})$:

$$J_n(\tilde{k}) = \{ \lambda(a, b, c, \tilde{k}) = \begin{bmatrix} a & -bk_1 & -bk_2 & \dots & bk_{n-1} \\ bk_1 & a - (1 - k_1^2)c & ck_1k_2 & \dots & ck_1k_{n-1} \\ bk_2 & ck_1k_2 & a - (1 - k_1^2)c & \dots & ck_2k_{n-1} \\ \vdots & & & & \\ bk_{n-1} & ck_1k_{n-1} & ck_2k_{n-1} & \dots & a - (1 - k_{n-1}^2)c \end{bmatrix} \}$$

$$(a, b) \neq (0, 0) \text{ and } a \neq c \} \subseteq GL(n, \mathbb{R})$$

$$\text{Remark : } \lambda(a, b, c, \tilde{k}) = M(\tilde{k})^T \lambda(a, b, c, \tilde{1}) M(\tilde{k})$$

where the 'base-point' in S^{n-2} is

$$\tilde{1} = (1, 0, \dots, 0) \in S^{n-2}$$

and

$$M(\tilde{k}) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & k_1 & k_2 & \dots & k_{n-1} \\ 0 & & & & \\ \vdots & & N & & \\ 0 & & & & \end{bmatrix}$$

N is any $(n-2) \times (n-1)$ matrix such that

$$M(\tilde{k}) \text{ is orthogonal i. e. } M(\tilde{k})^T = M(\tilde{k})^{-1}.$$

Lemma 3.1— $\det(\lambda(a, b, c, \tilde{k})) = (a^2 + b^2)(a - c)^{n-2}$.

PROOF : From the remark $\det(\lambda(a, b, c, \tilde{k})) = \det(\lambda(a, b, c, \tilde{1}))$
 $= (a^2 + b^2)(a - c)^{n-2}$.

[Also by direct calculation it can be proved that

$$\det(\lambda(a, b, c, \tilde{k})) = (a^2 + b^2)(a - c)^{n-2}].$$

Theorem 3.2— (i) $\tilde{J}_n(k)$ are commutative subgroups of $GL(n, \mathbb{R})$, (ii) Any two such subgroups are isomorphic to each other, (iii) $J_n(\tilde{k}) = J_n(-\tilde{k})$, (iv) for $\tilde{k}^{(1)} \neq \pm \tilde{k}^{(2)}$, $J_n(\tilde{k}^{(1)}) \cap J_n(\tilde{k}^{(2)}) = \{aI_n/a \neq 0, I_n \text{ is identity matrix}\}$.

PROOF : (i) follows from the fact that

$$\begin{aligned} \lambda(a_1, b_1, c_1, \tilde{k}). \lambda(a_2, b_2, c_2, \tilde{k}) &= \lambda(a_2, b_2, c_2, \tilde{k}). \lambda(a_1, b_1, c_1, \tilde{k}) \\ &= \lambda(a_1a_2 - b_1b_2, a_1b_2 + b_1a_2, a_1c_2 + c_1a_2 - c_1c_2 - b_1b_2, \tilde{k}) \end{aligned}$$

and

$$a_1 \neq c_1, a_2 \neq c_2 \text{ implies } a_1a_2 - b_1b_2 \neq a_1c_2 + c_1a_2 - c_1c_2 - b_1b_2.$$

And

$$\lambda(a, b, c, \tilde{k})^{-1} = \lambda\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}, \frac{-(ac + b^2)}{(a^2 + b^2)(a - c)}, \tilde{k}\right)$$

and

$$a \neq c \text{ implies } \frac{a}{a^2 + b^2} \neq \frac{-(ac + c^2)}{(a^2 + b^2)(a - c)}.$$

(ii) $\lambda(a, b, c, \tilde{k}^{(1)}) \mapsto \lambda(a, b, c, \tilde{k}^{(2)})$ gives an isomorphism.

(iii) follows from the definition of $J_n(\tilde{k})$

(iv) $aI_n = \lambda(a, 0, 0, \tilde{k}^{(1)}) = \lambda(a, 0, 0, \tilde{k}^{(2)})$ belongs to $J_n(\tilde{k}^{(1)}) \cap J_n(\tilde{k}^{(2)})$.

Conversely, let $\lambda(a_1, b_1, c_1, \tilde{k}^{(1)}) = \lambda(a_2, b_2, c_2, \tilde{k}^{(2)}) \in J_n(\tilde{k}^{(1)}) \cap J_n(\tilde{k}^{(2)})$.

If $b_1 \neq 0$ ($\therefore b_2 \neq 0$) comparing the terms of $\lambda(a_1, b_1, c_1, \tilde{k}^{(1)})$ and

$\lambda(a_2, b_2, c_2, \tilde{k}^{(2)})$ we get, either both $k_i^{(1)}$ and $k_i^{(2)}$ are zero or both

non zero and for non zero $k_i^{(1)}$ and $k_i^{(2)}$

$$\frac{\left(k_i^{(1)}\right)^2}{\left(k_i^{(2)}\right)^2} = \frac{\sum \left(k_i^{(1)}\right)^2}{\sum \left(k_i^{(2)}\right)^2} = 1$$

which implies $\tilde{k}^{(1)} = \pm \tilde{k}^{(2)}$.

Similarly if we assume $c_1 \neq 0$ ($\therefore c_2 \neq 0$) then for $k_i^{(1)} = 0$ if and only if $c_1 k_1^{(1)} k_i^{(1)}$

$= \dots = c_1 k_{n-1}^{(1)} k_i^{(1)} = 0$: if and only if $c_2 k_1^{(2)} k_i^{(2)} = \dots = c_2 k_{n-1}^{(2)} k_i^{(2)} = 0$; if and only if $k_i^{(2)} = 0$ and hence by the same argument $\tilde{k}^{(1)} = \pm \tilde{k}^{(2)}$. Thus for $\tilde{k}^{(1)} \neq \pm \tilde{k}^{(2)}$ $b_1 = 0 = c_1$ ($\therefore b_2 = 0 = c_2$).

Remark : Note that in virtue of Theorem 3.2 (iii) and 3.2 (iv) the distinct sub groups are parametrised by $\mathbb{P}^{n-2}(\mathbb{R})$ (real projective space).

Theorem 3.3—For any $\lambda (a, b, c, \tilde{k}) \in J_n(\tilde{k}) = J_n(-\tilde{k})$, ($\tilde{k} = (k_1, \dots, k_{n-1}) \in S^{n-2}$ and $n \geq 2$) and for any point p in {the 2-plane generated by the vectors $(1, 0, 0, \dots, 0)$ and $(0, k_1, \dots, k_{n-1})$ } \setminus \text{real line, (note, if } p = (x_0, x_1, \dots, x_{n-1}) \text{ then } x_1 : k_1 = x_2 : k_2 \equiv \dots \equiv x_{n-1} : k_{n-1}), \text{ there exists a Fueter mapping } f \equiv (f_0, \dots, f_{n-1}) \text{ [i. e. } \exists \text{ holo } \phi \text{ with } f = F_n(\phi)] \text{ such that}

$$d_p f \equiv [\text{Jac}(f)]_p = \lambda(a, b, c, \tilde{k}).$$

PROOF : $\lambda(a, b, c, \tilde{k}) \in J_n(\tilde{k})$.

Let

$$p = (p_0, p_1, k_1, \dots, k_{n-1})$$

we may assume $p_1 > 0$.

[Since if $p_1 < 0$ we can replace p_1 by $-p_1$, \tilde{k} by $-\tilde{k}$ and b by $-b$].

Consider the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \lambda^*(a, b)$.

There exists a complex analytic function $\phi = \xi + i\eta$ defined in a neighbourhood D of (p_0, p_1) in \mathbb{C} with $\eta(p_0, p_1) = (a - c)p_1$ such that

$$[\text{Jac}(\phi)]_{(p_0, p_1)} = \lambda^*(a, b).$$

Consider the Fueter function $f = F_n(\phi)$

$$\text{i. e. } f(x_0, \dots, x_{n-1}) = \xi(x_0, y) + \frac{e_1 x_1 + \dots + e_{n-1} x_{n-1}}{y} \eta(x_0, y)$$

where

$$y = [x_1^2 + \dots + x_{n-1}^2]^{1/2}.$$

Then,

$$[\text{Jac}(f)]_p = \begin{bmatrix} \partial_0 f_0 & \dots & \partial_{n-1} f_0 \\ \partial_0 f_1 & \dots & \partial_{n-1} f_1 \\ \dots & \dots & \dots \\ \partial_0 f_{n-1} & \dots & \partial_{n-1} f_{n-1} \end{bmatrix}_p$$

$$\begin{aligned}
 &= \begin{bmatrix} \eta_y & -\eta_{x_0} \frac{x_1}{y} & -\eta_{x_0} \frac{x_2}{y} & \dots & -\eta_{x_0} \frac{x_{n-1}}{y} \\ \eta_{x_0} \frac{x_1}{y} & \frac{x_1^2}{y^2} \left(\eta_y - \frac{\eta}{y} \right) + \frac{\eta}{y} & \frac{x_1 x_2}{y^2} \left(\eta_y - \frac{\eta}{y} \right) & \dots & \frac{x_1 x_{n-1}}{y^2} \left(\eta_y - \frac{\eta}{y} \right) \\ \eta_{x_0} \frac{x_2}{y} & \frac{x_1 x_2}{y} \left(\eta_y - \frac{\eta}{y} \right) & \frac{x_2^2}{y^2} \left(\eta_y - \frac{\eta}{y} \right) + \frac{\eta}{y} & \dots & \frac{x_2 x_{n-1}}{y^2} \left(\eta_y - \frac{\eta}{y} \right) \\ \dots & \dots & \dots & \dots & \dots \\ \eta_{x_0} \frac{x_{n-1}}{y} & \frac{x_1 x_{n-1}}{y^2} \left(\eta_y - \frac{\eta}{y} \right) & \frac{x_2 x_{n-1}}{y^2} \left(\eta_y - \frac{\eta}{y} \right) & \dots & \frac{x_{n-1}^2}{y^2} \left(\eta_y - \frac{\eta}{y} \right) + \frac{\eta}{y} \end{bmatrix}_p \\
 &= \lambda(a, b, c, \tilde{k}).
 \end{aligned}$$

Definition—From (7) and (9) it follows that the Fueter diffeomorphisms form pseudogroups. We call a n -dimensional C^∞ manifold a n -dimensional Fueter manifold if it is modelled on one such pseudogroup.

Thus, a Fueter manifold is a manifold with transition functions from the class of Fueter diffeomorphisms.

Theorem 3.4—(i) $\text{Det} [\text{Jac} f_n(\phi)]_{(x_0, x_1, \dots, x_{n-1})} = \left(\frac{\eta}{y} \right)^{n-2} [\text{det} (\text{Jac}(\phi))]_{(x_0, y)}$

where

$$\phi = \xi + i\eta y = \left[x_1^2 + \dots + x_{n-1}^2 \right]^{1/2} \neq 0.$$

(ii) Fueter manifold of even dimension (with coordinate charts $\psi : U \rightarrow \psi(U) \subseteq \mathbb{R}^n$) are orientable. Also those odd dimensional manifolds (we also called them Fueter manifolds) modelled on the pseudogroup of Fueter maps obtained from holomorphic maps of the type $\phi : D (\subseteq U) \rightarrow \phi(D) \subseteq U$ are orientable (for any dimension $n \geq 2$).

PROOF : (i) $\text{Det} [\text{Jac} F_n(\phi)]_{(x_0, x_1, \dots, x_{n-1})}$

$$\begin{aligned}
 &= \text{det} \left(\begin{bmatrix} \eta_y & -\eta_{x_0} \frac{x_1}{y} & \dots & \eta_{x_0} \frac{x_{n-1}}{y} \\ \eta_{x_0} \frac{x_1}{y} & \frac{x_1^2}{y^2} \left(\eta_y - \frac{\eta}{y} \right) + \frac{\eta}{y} & \dots & \frac{x_1 x_{n-1}}{y^2} \left(\eta_y - \frac{\eta}{y} \right) \\ \dots & \dots & \dots & \dots \\ \eta_{x_0} \frac{x_{n-1}}{y} & \frac{x_1 x_{n-1}}{y^2} \left(\eta_y - \frac{\eta}{y} \right) & \dots & \frac{x_{n-1}^2}{y^2} \left(\eta_y - \frac{\eta}{y} \right) + \frac{\eta}{y} \end{bmatrix}_{(x_0, y)} \right) \\
 &\left(\text{where } y = \left[x_1^2 + \dots + x_{n-1}^2 \right]^{1/2} \right) \\
 &= \text{det} (\lambda(a, b, c, \tilde{k})) \left[\text{where } k_i = \frac{x_i}{y} \quad i = 1, \dots, n-1, \right]
 \end{aligned}$$

$$\begin{aligned}\tilde{k} &= (k_1, \dots, k_{n-1}), b = \eta_{x_0}(x_0, y)] \\ &= (a^2 + b^2)(a - c)^{n-2} \left[a = \eta_y(x_0, y), c = \eta_y(x_0, y) - \frac{\eta(x_0, y)}{y} \right] \\ &= \left(\frac{n}{y} \right)^{n-2} \det ([\text{Jac}(\phi)]_{(x_0, y)}).\end{aligned}$$

(ii) From (i) if n is even and $\eta \neq 0$ then the determinant of the Jacobian of the Fueter transformation is positive. And therefore such manifolds are orientable.

We observe that $X \equiv (x^1, x^2, x^3): \lambda(a, b, c, \tilde{k}) \mapsto (a, b, c)$ defines a global C^∞ structure on $J_n(\tilde{k})$. The multiplication in $J_n(\tilde{k})$ is equivalent to the mapping! $\mathbb{R}^6 \equiv \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $((a_1, b_1, c_1), (a_2, b_2, c_2)) \mapsto (a_1 a_2 - b_1 b_2, a_1 b_2 + b_1 a_2, a_1 c_2 + c_1 a_2 - b_1 c_2 - b_2 c_1)$ which is C^∞ , and $\lambda(a, b, c, \tilde{k}) \mapsto \lambda(a, b, c, \tilde{k})^{-1}$ is equivalent to the map:

$$' \mathbb{R}^3 \rightarrow ' \mathbb{R}^3 \text{ as } (a, b, c) \mapsto \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}, \frac{-(ac + b^2)}{(a^2 + b^2)(a - c)} \right)$$

which is also C_∞ . This shows directly that $J_n(\tilde{k})$ is a 3-dimensional Lie subgroup of $GL(n, \mathbb{R})$.

Remark: The isomorphisms defined in Th. 3.2 are actually Lie group isomorphisms.

Remark: We can calculate explicitly the Lie algebra $L(J_n(\tilde{k}))$ (the Lie algebra of all left invariant vector fields) $T_e(J_n(\tilde{k}))$ (the tangent space of $J_n(\tilde{k})$ at the identity ($e = \lambda(1, 0, 0, \tilde{k})$)) = $\langle e_1, e_2, e_3 : e_i = \left(\frac{\partial}{\partial x^i} \right)_e \rangle$

Let $\mathcal{L}(J_n(\tilde{k}))$ be all the left invariant vector fields. Then

$$\mathcal{L}(J_n(\tilde{k})) = \langle X_1, X_2, X_3 \mid X: \lambda \rightarrow L_\lambda * \left(\frac{\partial}{\partial x^i} \right)_e; i = 1, 2, 3 \rangle.$$

Since $J_n(k)$ is commutative than of course all Lie brackets vanish. Also by some simple calculations

$$X_1 = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3}$$

[these are a basis for the left-invariant vector fields]

$$X_2 = x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} - x^3 \frac{\partial}{\partial x^3}$$

$$X_3 = (x^1 - x^3) \frac{\partial}{\partial x^3}.$$

And,

$$[X_1, X_2] = [X_1, X_3] = [X_2, X_3] = 0.$$

For higher dimensional Fueter transforms $(F_n^{(l)})$ the Jacobians form families of non-commutative Lie subgroups (of dimension $2l^2 + l$) in $GL(\eta l, \mathbb{R})$. The Lie algebra of any one of these subgroups is non trivial and has been explicitly computable, (details are available with the author).

For 2-dimensional Fueter transformation $F_n^{(2)}$ the Jacobians are of the form

$$\lambda(\sigma, b, c, d, e, f, g, h, p, q, k, m) = \begin{pmatrix} a & c & k_1 e & -m_1 g & -k_2 e & m_2^2 g & \dots & -k_{n-1} e & -m_{n-1} g \\ b & d & -k_1 f & -m_1 h & -k_2 f & -m_2 h & \dots & -k_{n-1} f & -m_{n-1} h \\ l k_1 g k_1 & a - (1 - k_1^2) p & k_1 m_1 c & k_1 m_1 c & k_1 k_2 p & k_1 m_2 c & \dots & k_1 k_{n-1} p & k_1 k_{n-1} c \\ m_1 f & m_1 h & k_1 m_1 p & d - (1 - m_1^2) q & m_1 k_2 b & m_1 m_2 q & \dots & m_1 k_{n-1} b & m_1 m_{n-1} q \\ k_2 e & k_2 g & k_1 k_2 b & m_1 k_2 c & a - (1 - k_2^2) c & k_1 m_2 c & \dots & k_2 k_{n-1} p & c k_2 m_{n-1} \\ m_2 f & k_2 h & k_1 m_2 b & m_1 m_2 q & k_2 m_2 b & d - (1 - m_2^2) q & \dots & m_2 k_{n-1} b & m_2 m_{n-1} q \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ k_{n-1} e & k_{n-1} g & k_1 k_{n-1} p & m_1 k_{n-1} c & \dots & \dots & \dots & a - (1 - k_{n-1}^2) p & c k_{n-1} m_{n-1} \\ m_{n-1} f & m_{n-1} h & k_2 m_{n-1} b & m_2 m_{n-1} q & \dots & \dots & \dots & \dots & k_{n-1} m_{n-1} b & d - (1 - m_{n-1}^2) q \end{pmatrix} \dots (23)$$

(where $\tilde{k}, \tilde{m} \in s^{n-2}$)

with $\det (\lambda (a, b, c, d, e, f, g, h, p, q, \tilde{k}, \tilde{m}))$.

$$= (ad - bc)^2 + (eh - gf)^2 (a - p)^{n-2} (b - q)^{n-2}. \quad \dots(24)$$

And for $(\tilde{k}, \tilde{m}) \in S^{n-2} \times S^{n-2}$

$$J_n (\tilde{k}, \tilde{m}) = \{ \lambda (a, b, c, d, e, f, g, h, p, q, \tilde{k}, \tilde{m}) : a \neq p, b \neq q \\ (ad, eh) \neq (bc, gf) \} \subseteq GL(2n, \mathbb{R})$$

is a non-commutative Lie group with multiplication:

$$\lambda (a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1, p_1, q_1, \tilde{k}, \tilde{m}) \cdot \lambda (a_2, b_2, c_2, d_2, e_2, f_2, g_2, h_2, p_2, q_2, \tilde{k}, \tilde{m}) \\ = \lambda \left(\begin{bmatrix} a_1 & a_2 \\ +c_1 & b_2 \\ -e_1 & e_2 \\ -g_1 & f_2 \end{bmatrix}, \begin{bmatrix} b_1 & a_2 \\ +d_1 & b_2 \\ -f_1 & e_2 \\ -h_1 & f_2 \end{bmatrix}, \begin{bmatrix} a_1 & c_2 \\ +c_1 & d_2 \\ -e_1 & g_2 \\ -g_1 & h_2 \end{bmatrix}, \begin{bmatrix} b_1 & c_2 \\ +d_1 & d_2 \\ -f_1 & g_2 \\ -h_1 & h_2 \end{bmatrix} \right. \\ , \begin{bmatrix} a_1 & e_2 \\ +c_1 & f_2 \\ +c_1 & a_2 \\ +g_1 & b_2 \end{bmatrix}, \begin{bmatrix} b_1 & e_2 \\ +d_1 & f_2 \\ +f_1 & a_2 \\ +h_1 & b_2 \end{bmatrix}, \begin{bmatrix} a_1 & g_2 \\ +c_1 & h_2 \\ +e_1 & c_2 \\ +g_1 & d_2 \end{bmatrix}, \begin{bmatrix} b_1 & g_2 \\ +d_1 & h_2 \\ +f_1 & c_2 \\ +h_1 & d_2 \end{bmatrix} \\ \left. \begin{bmatrix} c_1 & b_2 \\ -e_1 & e_2 \\ -g_1 & f_2 \\ +a_1 & p_2 \\ +p_1 & a_2 \\ -p_1 & p_2 \end{bmatrix}, \begin{bmatrix} b_1 & c_2 \\ -f_1 & g_2 \\ -h_1 & h_2 \\ +d_1 & q_2 \\ +q_1 & d_2 \\ -q_1 & q_2 \end{bmatrix}, \begin{matrix} \tilde{k} \\ \tilde{m} \end{matrix} \right)$$

...(25)

And the Lie Algebra of any of the Lie groups is

$$L(J_n(\tilde{k}, \tilde{m})) = \langle X_1, \dots, X_{10}, [,] \rangle \text{ where}$$

$$X_1 = x^1 \partial_1 + x^2 \partial_2 + x^5 \partial_5 + x^6 \partial_6 + x^9 \partial_9,$$

$$X_2 = x^3 \partial_1 + x^4 \partial_2 + x^7 \partial_5 + x^8 \partial_6 + x^3 \partial_9,$$

$$X_3 = x^1 \partial_1 + x^2 \partial_4 + x^5 \partial_7 + x^6 \partial_8 + x^2 \partial_{10},$$

$$X_4 = x^3 \partial_3 + x^4 \partial_4 + x^7 \partial_7 + x^8 \partial_8 + x^{10} \partial_{10},$$

$$X_5 = -x^5 \partial_1 - x^6 \partial_2 + x^1 \partial_5 + x^2 \partial_6 - x^5 \partial_9,$$

$$X_6 = -x^7 \partial_1 - x^8 \partial_2 + x^3 \partial_5 + x^4 \partial_6 - x^7 \partial_9,$$

$$X_7 = -x^5 \partial_3 - x^6 \partial_4 + x^1 \partial_7 + x^2 \partial_8 - x^6 \partial_{10},$$

$$\begin{aligned} X_9 &= -x^4 \partial_3 - x^8 \partial_4 + x^3 \partial_7 + x^4 \partial_8 - x^8 \partial_{10}, \\ X_9 &= (x^1 - x^9) \partial_9, \quad X_{10} = (x^4 - x^{10}) \partial_{10} \end{aligned} \quad \dots(26)$$

and

$$\begin{aligned} [X_1, X_3] &= [X_3, X_4] = [X_7, X_5] = [X_8, X_7] = X_3, \\ [X_6, X_1] &= [X_8, X_2] = [X_4, X_6] = [X_2, X_6] = X_6, \\ [X_1, X_7] &= [X_5, X_3] = [X_3, X_8] = [X_7, x_4] = X_7, \\ [X_4, X_2] &= [X_6, X_8] = X_2, [X_2, X_7] = [X_6, X_3] = X_8 - X_5, \\ [X_6, X_7] &= X_1, [X_2, X_3] = X_4 + X_{10} - X_1 - X_9, \\ [X_i, X_j] &= 0 \end{aligned} \quad \dots(27)$$

where $[X_i, X_j]$ or $[X_j, X_i]$ are not amongst the above.

4. CONNECTION BETWEEN FUETER AND REGULAR MAPS

Imaeda and Imaeda⁶ had defined some generalizations of analytic functions of an octonionic variable which are similar to those for quaternionic variables defined and discussed by Fueter⁵. Imaeda and Imaeda⁷ also generalized these concepts over variables from more general algebras. The Fueter maps which are discussed in Section 1 and 2 and the regular maps defined by Imaeda and Imaeda actually from disjoint classes (except for the constant maps, being members of both). However, there is some connection between these two classes which we are going to discuss.

Definition —(due to Imaeda and Imaeda, of regular functions). A map $f \equiv (f_0, \dots, f_{n-1}) \equiv e_0 f_0 + \dots + e_{n-1} f_{n-1}: \tilde{D} (\subseteq \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is called left regular (resp. right regular) if $Df = 0$ (resp. $fD = 0$) where $D = \sum_{j=0}^{n-1} e_j \partial_j$ and $e_j e_k + e_k e_j = -2\delta_{j,k}$.

Proposition 4.1—A map f both Fueter and regular in the sense of Imaeda and Imaeda if and only if it is a real constant.

PROOF: Let $f: \tilde{D} (\subseteq \mathbb{R}^n) \rightarrow \mathbb{R}^n$ be both left regular and Fueter \tilde{D} i.e., $Df(x_0, \dots, x_{n-1}) = 0$ for $(x_0, \dots, x_{n-1}) \in \tilde{D}$ and there exists holomorphic map $\phi = \xi + i\eta$ on the relevant domain such that $f = F_n(\phi)$ on \tilde{D} . Equivalently

$$\left(\delta_0 + \sum_{j=1}^{n-1} e_j \partial_j \right) \left(\xi(x_0, y) + \sum_{j=1}^{n-1} \frac{e_j x_j}{y} \eta(x_0, y) \right) = 0$$

(where $y = \left(x_1^2 + \dots + x_{n-1}^2 \right)^{1/2}$) which gives by using Cauchy-Riemann relation between ξ and η , $\frac{(n-2)\eta(x_0, y)}{y} = 0$.

Therefore for $n > 2$, $\eta = 0$, which implies ϕ and therefore f are constant maps which constant real values. Similarly for right regularity.

For even dimensions one can generate regular maps in the sense of Imaeda from Fueter maps. For interest's sake we state this connection below (see Imaeda Imaeda⁶).

Proposition 4.2—If $f: \widetilde{D} (\subseteq \mathbb{R}^{2s}) \rightarrow \mathbb{R}^{2s}$ is a Fueter map then $\square^{-1} f$ is both left and right regular. (where $\square = D \bar{D} = \sum_{j=0}^{2s-1} \delta_j^2$).

Remark 1: Note that for the trivial 2-dimensional case Fueter maps and Imaeda's regular maps all coincide with usual holomorphic maps.

Remark 2: As a general reference for previous work in this and allied areas we refer to Brackx *et al.*².

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NONLINEAR COMPLEMENTARITY PROBLEM OF MATHEMATICAL PROGRAMMING IN BANACH SPACE

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In this paper short proofs of some existence theorems on nonlinear complementarity problem of mathematical programming in a reflexive real Banach space are given for arbitrary closed convex cones under much weaker assumptions (in the absence of the boundedness) on the operator.

INTRODUCTION

Let B be a reflexive real Banach space and let B^* be its dual. Let the value of $u \in B^*$ at $x \in B$ be denoted by (u, x) . Let C be a closed convex cone in B with the vertex at 0. The polar of C is the cone C^* , defined by

$$C^* = \{u \in B^* : (u, x) \geq 0 \text{ for each } x \in C\}.$$

For each $r \geq 0$ we write

$$D_r = \{x \in C : \|x\| \leq r\}$$

$$D_r^0 = \{x \in C : \|x\| < r\}$$

$$S_r = \{x \in C : \|x\| = r\}.$$

A mapping $T : C \rightarrow B^*$ is said to be monotone if $(Tx - Ty, x - y) \geq 0$ for all $x, y \in C$ and strictly monotone if strict inequality holds whenever $x \neq y$. T is said to be α -monotone if there is a strictly increasing function $\alpha : [0, \infty) \rightarrow [0, \infty)$ with $\alpha(0) = 0$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$ such that $(Tx - Ty, x - y) \geq \alpha(\|x - y\|)$ for all $x, y \in C$. In particular, T is strongly monotone if $\alpha(r) = kr$ for some constant $k > 0$. T is said to be coercive, on C if $\frac{(Tx, x)}{\|x\|} \rightarrow \infty$ as $\|x\| \rightarrow \infty$ for $x \in C$. T is called hemicontinuous on C if for all $x, y \in C$, the map $t \rightarrow T((1-t)x + ty)$ of $[0, 1]$ to B^* is continuous, when B^* is endowed with the weak* topology. T is called bounded if T maps bounded subsets of C into bounded subsets of B^* .

The purpose of this paper is to give short proofs of some existence theorems on nonlinear complementarity problem of mathematical programming in Banach space for arbitrary closed convex cones under much weaker assumptions on the operator (in the absence of the boundedness of the operator).

MAIN RESULT

We prove the following result :

Theorem—Let $T: C \rightarrow B^*$ be hemi-continuous and monotone. Then there exists x_0 such that

$$x_0 \in C, Tx_0 \in C^* \text{ and } (Tx_0, x_0) = 0 \quad \dots(1)$$

under each of the following conditions :

- (a) T is coercive (or in particular T is α -monotone),
- (b) $T0 \in C^*$ (or in particular $T0 = 0$),
- (c) For atleast one $r > 0$, there exists an $u \in D_r^0$ with $(Tx, x - u) > 0$ for all $x \in S_r$.

This theorem under the conditions (a) and (b) was respectively obtained by Bazaraa *et al.*¹ and Nanda and Nanda⁵ under an additional assumption (i. e., boundedness) on the operator T . It should further be noted that the above theorem under hemicontinuity and α -monotonicity is contained in the result of Luna³ also.

Note that the result that there exists an x_0 with satisfies (1) may not hold only under the assumption of hemicontinuity and monotonicity of the operator T . For example, take $B = R$, $C = \{x : x \geq 0\}$, so that $C = C^*$. Let $T: C \rightarrow R$ be defined by

$$Tx = -\frac{1}{1+x}.$$

In this case T is hemicontinuous and strictly monotone. $(Tx, x) = 0$ implies $x = 0$, but $T0 = -1 \notin C^*$.

We need the following result of Browder² in order to prove our theorems. See also Mosco⁴.

Proposition—Let T be a monotone and hemicontinuous map of a closed convex set K in B , with $0 \in K$, into B^* , and if K is not bounded, let T be coercive on K . Then there is an $x_0 \in K$ such that

$$(Tx_0, y - x_0) \geq 0 \text{ for all } y \in K. \quad \dots(2)$$

The inequalities of the form (2) are called variational inequalities.

PROOF OF THE THEOREM

Proof of (a)—Since T is hemicontinuous, monotone and coercive and C is closed and convex, it follows from the Proposition that there exists an $x_0 \in C$ such that

$$(Tx_0, y - x_0) \geq 0 \text{ for all } y \in C.$$

Since $0 \in C$, $(Tx_0, x_0) \leq 0$. Also since C is a cone, $2x_0 \in C$ and hence $(Tx_0, x_0) \geq 0$. Therefore, $(Tx_0, x_0) = 0$. Now $Tx_0 \in C^*$. For otherwise there would exist $y_0 \in C$ such that $(Tx_0, y_0) < 0$. Then we have

$$0 > (Tx_0, y_0) \geq (Tx_0, x_0) = 0.$$

This contradiction completes the proof.

Proof of (b)—Note that for each $r \geq 0$, D_r is a nonempty closed convex set in C with $0 \in D_r$. Therefore it follows from the Proposition that for each $r \geq 0$, there exists a unique $x_r \in D_r$ such that

$$(Tx_r, y - x_r) \geq 0 \text{ for all } y \in D_r. \quad \dots(3)$$

Since $0 \in D_r$, it follows that $(Tx_r, x_r) \leq 0$. Since T is monotone we have

$$(Tx_r - T0, x_r) \geq 0$$

and further if $T0 \in C^*$ or $T0 = 0$ we obtain

$$(Tx_r, x_r) \geq 0.$$

Therefore $(Tx_r, x_r) = 0$ and hence x_r is a solution of (1) for each $r \geq 0$.

Proof of (c)—As in the above case for each $r > 0$ there exists an $x_r \in D_r$ such that (3) holds and consequently $(Tx_r, x_r) \leq 0$. If there is some $r > 0$ such that $x_r \in D_r^0$, then there is some $\lambda > 1$ such that $\lambda x_r \in S_r \subset D_r$. Then we have from (3) that $(Tx_r, x_r) \leq \lambda (Tx_r, x_r)$. Since $(Tx_r, x_r) \leq 0$, this is impossible unless $(Tx_r, x_r) = 0$. Thus x_r is a solution of (1). On the other hand, let $x_r \in S_r$ for all $r > 0$. Then by the hypothesis for atleast one $r > 0$, there is a $u \in D_r^0$ such that $(Tx_r, x - u) \geq 0$ for all $x \in S_r$. Therefore for that r ,

$$(Tx_r, x_r - u) \geq 0. \quad \dots(4)$$

From (3) and (4) it follows that

$$(Tx_r, y - u) \geq 0 \text{ for all } y \in D_r.$$

Let $Z \in C$. Write $w = \lambda Z + (1 - \lambda)u$, $0 < \lambda < 1$.

We can choose λ sufficiently small so that w lies in D_r . Then

$$0 \leq (Tx_r, w - u) = \lambda (Tx_r, Z - u).$$

Therefore

$$(Tx_r, Z - u) \geq 0 \text{ for all } Z \in C. \quad \dots(5)$$

Since $u \in D_r^0$ it follows from (3) that

$$(Tx_r, u - x_r) \geq 0. \quad \dots(6)$$

From (5) and (6) we obtain

$$(Tx_r, Z - x_r) \geq 0 \text{ for all } Z \in C. \quad \dots(7)$$

Taking $Z = \lambda x_r$, $\lambda > 1$, in (7) we get $(Tx_r, x_r) > 0$. Since $(Tx_r, x_r) \leq 0$ we obtain $(Tx_r, x_r) = 0$. Also since

$$(Tx_r, Z) \geq (Tx_r, x_r) = 0 \text{ for all } Z \in C, Tx_r \in C^*.$$

Thus x_r is a solution of (1) and this completes the proof.

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ON SEMI-METRIZABLE-CLOSED SPACES

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This paper considers the class of locally semi-metrizable-closed spaces. In particular, it studies one-point feeble-compactifications of such spaces. The projective objects in the category of semi-metrizable-closed spaces and continuous maps are shown to be the finite spaces.

The class of 'absolutely closed spaces', also called ' H -closed spaces', was first introduced by Alexandroff and Urysohn. A Hausdorff space is said to be H -closed if and only if it is closed in every embedding into a Hausdorff space. Porter⁸ studied in detail locally H -closed spaces and proved that the one-point H -closed extensions of a locally H -closed space are not unique and that there exists a projective maximum and a projective minimum among them. The study of minimal first countable spaces was carried out by Stephenson^{12,13} and Porter⁹. Further the related notion of $H(1)$ -closed spaces and other allied spaces were studied by Daniel Thanapalan and Raghavan^{3,4} and Raghavan¹⁰. We define a topological space to be a ' $H(1)$ -space' if and only if it is Hausdorff and first countable. A $H(1)$ -space is called ' $H(1)$ -closed' if and only if it is closed in every embedding into a $H(1)$ -space. A complete survey of results in this direction is given in Berri *et al.*².

Semi-metrizable-closed and minimal semi-metrizable spaces were investigated by Stephenson¹⁴. He gave several characterizations of such spaces. In this paper locally semi-metrizable-closed spaces are investigated in detail. In particular we consider one-point feeble-compactifications of a locally semi-metrizable-closed space. As an application we prove that the projective objects in the category of semi-metrizable-closed spaces and continuous maps are finite spaces.

If (X, τ) is a topological space and $B \subset A \subset X$, we write $\text{cl } B$ for the closure of B in (X, τ) , $\text{int } B$ for the interior of B in (X, τ) , τ/A for the relative topology of τ on A and $\text{cl}_A B$ and $\text{int}_A B$ for the closure and interior respectively of B in $(A, \tau/A)$. We also use $\tau(X)$ to denote the topology on X . N stands for the set of positive integers, 'nbd' means neighbourhood, and \square denotes the end of proof.

1. LOCALLY SM-CLOSED SPACES

Definition 1.1—A topological space X is semi-metrizable if there exists a mapping $d : X \times X \rightarrow [0, \infty)$ such that (i) for all $x, y \in X$ $d(x, y) = d(y, x)$ and $d(x, y)$

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$= 0$ if and only if $x = y$; (ii) a subset V of X is open if and only if for each $x \in V$ there exists $n \in N$ such that $B(n, x) = \{y \in X \mid d(n, y) < 1/n\} \subset V$ and (iii) each $B(n, x)$ is a nbd of x .

Definition 1.2—A semi-metrizable Hausdorff space X is called semi-metrizable-closed ($=SM$ -closed) if and only if X is closed in every semi-metrizable Hausdorff space into which it is embedded.

Definitions 1.3—(a) A collection C of subsets of a topological space (X, τ) is called an almost cover of $A (\subset X)$ if and only if $A \subset \bigcup \{cl U \mid U \in C\}$.

(b) A topological space X is called feebly compact if and only if every countable open filter base F has nonempty adherence [i. e., $\text{adh}(F) = \bigcap \{cl U \mid U \in F\} \neq \emptyset$].

The equivalence of (i) and (ii) in the following theorem is from Stephenson¹⁴; the equivalence of (ii) and (iii) is clear.

Theorem 1.4—Let (X, τ) be a semi-metrizable Hausdorff space. Then the following are equivalent :

- (i) X is SM -closed;
- (ii) X is feebly compact;
- (iii) Every countable open cover of X has a finite almost subcover.

Definitions 1.5—Let (X, τ) be a Hausdorff topological space.

(a) A subset A of X is called 'feebly compact relative to X ' if and only if every countable open cover of A in X has a finite almost subcover.

(b) A subset A of X is called 'a feebly compact set' if and only if $(A, \tau/A)$ is feebly compact.

(c) X is called 'locally feebly compact' if and only if every point of X has an open nbd whose closure in X is feebly compact relative to X .

Feeble-compactness and other related notions are defined for Hausdorff spaces which need not be semi-metrizable, whereas SM -closedness and other related notions are defined only for semi-metrizable Hausdorff spaces. Indeed, in semi-metrizable spaces the concepts SM -closedness relative to X and feeble compactness relative to X are identical and a semi-metrizable Hausdorff space is locally SM -closed if and only if it is locally feebly compact.

Thus we work with Hausdorff spaces and study the one-point feeble compactifications of locally feebly compact space, the appropriate modifications, when the base space is semi-metrizable, being clear. The following theorem is easy to verify.

Theorem 1.6—Let X be Hausdorff space and $A \subseteq B \subseteq X$.

(a) If X is first countable and A is feebly compact relative to X , then A is closed in X .

(b) If A is feebly compact, then A is feebly compact relative to X .

(c) If A is regular closed in X and B is feebly compact relative to X , then A is feebly compact.

Consequently we have the following result.

Theorem 1.7—Let X be semi-metrizable Hausdorff space and $A \subset B \subset X$.

(a) If A is SM closed relative to X , then A is closed in X , then A is closed in X .

(b) If A is SM-closed, then A is SM-closed relative to X .

(c) If A is regular-closed and B is SM-closed relative to X , then A is SM-closed.

It is clear that a semi-metrizable Hausdorff space is locally SM-closed if and only if each point has either an open nbd whose closure is SM-closed relative to X or an open nbd whose closure is a SM-closed set or a nbd which is SM-closed relative to X .

2. ONE-POINT EXTENSIONS

In this section we define and study one-point extensions of locally feebly compact Hausdorff spaces with or without semi-metrizability of the base space. These extensions are Hausdorff and they may or may not be semi-metrizable.

Definition 2.1—Let (X, τ) be a (semi-metrizable) Hausdorff space. (Y, σ) is called a (one-point SM-closed extension) one-point Hausdorff feeble-compactification of (X, τ) if and only if $X \subset Y$, $\sigma|_X = \tau$, $\sigma(X) = Y$, (Y, σ) is (SM-closed) feebly compact and Hausdorff and $Y - X$ is a singleton.

Proposition 2.2—Let (X, τ) be a semi-metrizable Hausdorff space. If (Y, σ) is a one-point Hausdorff feeble-compactification of (X, τ) , then X is open in Y and (X, τ) is locally SM-closed.

PROOF : Let $Y - X = \{\pi\}$. Since (Y, σ) is Hausdorff, $\{\pi\}$ is σ -closed and hence X is σ -open in Y . Further for each $x \in X$, there exists a σ -open nbd U of x such that $\pi \notin \text{cl}_Y U$. Since Y is feebly compact, $\text{cl}_Y U$ is feebly compact (Theorem 14 of Bagley *et al.*¹). Since X is a semi-metrizable Hausdorff space it is locally SM-closed.

We now give the definition of “projectively smaller”, a relation which is used to compare the extensions of a given space. This well-known notion is due to Banaschewski.

Definitions 2.3—Let (Y, σ) and (Z, ρ) be extensions of topological space (X, τ) . The extension (Z, ρ) is called projectively smaller than (Y, σ) if and only if there exists a continuous function $f : (Y, \sigma) \rightarrow (Z, \rho)$ such that $f(x) = x$ for all $x \in X$. (Y, σ) and (Z, ρ) are called isomorphic if and only if there is a homeomorphism $h : (Y, \sigma) \rightarrow (Z, \rho)$ which leaves X pointwise fixed. If \mathcal{E} is a family of extensions of a space (X, τ) , then (Y, σ) in \mathcal{E} is called a projective minimum in \mathcal{E} if and only if (Y, σ) is

projectively smaller than every member of \mathcal{E} . The concepts of "projectively larger than" and "projective maximum" in \mathcal{E} are defined dually.

In the following theorem we consider the one-point Hausdorff feeble-compactifications of a locally feebly compact Hausdorff space.

Theorem 2.4—Let X be a locally feebly compact Hausdorff space which is not feebly compact. There are spaces X^* and X^\dagger such that

(i) X^* and X^\dagger are both one-point Hausdorff feeble-compactifications where $X^* - X = X^\dagger - X = \{\pi\}$,

(ii) $\tau(X^*) \subset \tau(X^\dagger)$,

(iii) $\tau(X^*) = \tau(X) \cup \{\{\pi\} \cup V \mid V \in \alpha\}$ where α is the open filter generated by $\{U \mid U \in \tau(X) \text{ and } X - U \text{ is feebly compact relative to } X\}$,

(iv) $\tau(X^\dagger) = \tau(X) \cup \{\{\pi\} \cup V \mid V \in \tau(X) \text{ and } X - \text{int cl } V \text{ is feebly compact}\}$,

(v) X^\dagger is projective maximum among all the one-point feeble-compactifications of X ,

(vi) If X admits a one point feeble compactification which is first countable then X^* is projectively smaller than every one-point feeble-compactification that is first countable.

PROOF : Let us briefly outline the proof; one may also refer to the proof of the corresponding one in Porter⁸. If $\beta = \{U \mid U \in \tau(X) \text{ and } X - U \text{ is feebly compact relative to } X\}$, then β is an open filter base generating the open filter α . Let $\gamma = \{V \mid V \in \tau(X) \text{ and } X - \text{int cl } V \text{ is feebly compact}\}$. Then γ is an open filter. It is easily verified that $\tau(X)$ and $\tau(X^\#)$ are Hausdorff topologies and $\tau(X^*) \subset \tau(X^\#)$, since $\alpha \subset \gamma$.

If $\mathcal{C} = \{G_i \mid i \in N\}$ be a countable $\tau(X^*)$ -open over of X^* , then there exists a $\tau(X^*)$ -open nbd U of π and $k \in N$ such that $\{\pi\} \cup U \subset G_k \in \mathcal{C}$ and $X - U$ is feebly compact relative to X . Clearly $\{G_i \cap X \mid i \in N\}$ is a countable open cover of $X - U$ so that $X - U \subset \bigcup_{j=1}^r \text{cl}(G_{n_j} \cap X) \subset \text{cl}_*(G_{n_j})$ where cl_* refers to closure taken with respect to $\tau(X^*)$. Thus $X^* = \text{cl}_*(G_k) \cup (\bigcup_{j=1}^r \text{cl}_*(G_{n_j}))$ so that X^* is feebly compact.

If Z is any first countable one-point feeble-compactification of X and $Z - X = \{\eta\}$, then defining $g : Z \rightarrow X^*$ by setting $g(x) = x$ for all $x \in X$ and $g(\eta) = \pi$, we get a continuous map. Indeed g is continuous at η . For $g^{-1}(\{\pi\} \cup U) = \{\eta\} \cup U$ and if $X - U$ is feebly compact relative to X , it is feebly compact relative to Z , so that, by Theorem 1.6 (a), it is closed in Z .

The other parts of the proof follow along similar lines.

It must be noted that every locally feebly compact Hausdorff space admits a one-point Hausdorff feeble compactification. On the other hand not every semi-metrizable Hausdorff space which is locally SM-closed admits a one-point SM-closed extension. This is precisely the situation of Example 2.6 that follows. Further, there are nice examples of locally SM-closed spaces admitting one-point SM-closed extension as we see in Example 2.7 below.

Theorem 2.5—Let X be a semi-metrizable Hausdorff locally SM-closed space which is not SM-closed. Let $FC(X)$ be the set of all one-point Hausdorff feebly-compactifications and $SM(X)$ the set of all one-point SM-closed extensions of X . Then $FC(X) \neq \phi$ and $SM(X) \subset FC(X)$. Further, if $SM(X)$ is non-empty then $\tau(X^*)$ of Theorem 2.4 is projectively smaller than every element in $SM(X)$.

Example 2.6—We have proved in Theorem 2.4 that $(X^*, \tau(X^*))$ is a one-point feeble compactification which is Hausdorff. However, it should be noted that $(X^*, \tau(X^*))$ need not be SM-closed. If we take X to be an uncountable set, say, R , the set of all real numbers and τ to be the discrete topology on R , then $(R^*, \tau(R^*))$ is feebly compact by Theorem 2.4; indeed $(R^*, \tau(R^*))$ is the one-point compactification of R , but not semi-metrizable. Moreover, in this case $\tau(R^*) = \tau(R^\#)$. Thus $(R, \tau(R))$ does not have a one-point SM-closed extension; for, if $(R^*, \sigma(R^*))$ is SM-closed, then $\tau(R^*) \subset \sigma(R^*) \subset \tau(R^\#)$.

Example 2.7—There exist locally SM-closed spaces which admit one-point SM-closed extensions. Consider the following example given by Stephenson¹⁴ (Example 6). Let

$$X = N \cup \{n \pm 1/m = n, m \in N, m > 2\} \cup \{\pm \pi\}$$

where N is the set of positive integers. Define $d: X \times X \rightarrow [0, \infty)$ as follows:

$$d(x, y) = d(y, x) = \begin{cases} 0 & \text{if } x = y \\ |x - y| & \text{if } x, y \notin \{\pm \pi\} \\ 1 & \text{if } x \in N \text{ and } y \in \{\pm \pi\}, \text{ or} \\ & \text{if } x = n + 1/m \text{ and } y = -\pi, \text{ or} \\ & \text{if } x = n - 1/m \text{ and } y = +\pi, \text{ or} \\ & \text{if } x = \pi \text{ and } y = -\pi \\ & \text{where } m, n \in N \text{ and } m > 2 \\ 1/n & \text{if } x = n + 1/m \text{ and } y = \pi, \text{ or} \\ & \text{if } x = n - 1/m \text{ and } y = -\pi \\ & \text{where } m, n \in N \text{ and } m > 2. \end{cases}$$

Here a subset V is open if and only if for each point $x \in V$ there exists an $\epsilon > 0$ with $\{y \mid d(x, y) < \epsilon\} \subset V$. Then d is a semi-metric for the space. Stephenson pointed out that this space X is minimal symmetrizable. Since it is semi-metrizable, it is also minimal semi-metrizable. Let us take

$$Y = N \cup \{n + 1/m \mid n, m \in N, m > 2\} \cup \{\pi\}.$$

Y is a subspace of X and Y is a SM-closed space. Now let us take

$$Z = N \cup \{n + 1/m \mid n, m \in N, m > 2\}.$$

Z is a locally SM-closed space and $Y = Z \cup \{\pi\}$ is a one-point SM-closed extension of Z . (There are other semi-metrizable locally SM-closed spaces which admit one-point SM-closed extensions; but what is unique about the above example is that it is *not* regular.)

3. FURTHER RESULTS

In this section we determine the projective objects in the category of SM-closed spaces and continuous maps. For the definition of projective objects in a category one may either refer to Herrlich⁶ or Raghavan and Reilly¹¹.

Theorem 3.1—In the category of SM-closed spaces and continuous maps, the projective objects are finite spaces.

Also in the category of feebly compact spaces and continuous maps the projective objects are finite spaces.

PROOF : That X is projective when X is finite is obvious.

Conversely, if (X, τ) is projective, let us show that X is discrete so that it is finite.

Suppose X is not discrete. Then there exists $a \in X$ such that a is not isolated. Let $Y = X - \{a\}$. Let π be an abstract point not in $Y \times N^*$, where $N^* = N \cup \{\omega\}$, the one-point compactification of the discrete space N of positive integers. Let $A = (Y \times N^*) \cup \{\pi\}$. Let us introduce a topology ρ on A as follows :

- (i) If $z \in Y \times N^*$ then the set of all open nbds of z in the product topology of Y and N^* is taken as a fundamental system of ρ -open nbds of z .
- (ii) Sets of the form $(O \times N) \cup \{\pi\}$ where O is a deleted open nbd of a in X form a fundamental system of ρ -open nbds of π .

(By deleted open nbd O of a , we mean that $O \cup \{a\}$ is open in X and $a \notin O$.) (i) and (ii) completely specify the topology ρ on A .

We can take the semi-metric on N^* as follows :

$$d_2(m, n) = |1/m - 1/n| \text{ for all } m, n \in N$$

$$d_2(\omega, m) = d_2(m, \omega) = 1/m \text{ for all } m \in N$$

$$d_2(\omega, \omega) = 0.$$

If d_1 is the semi-metric on X , d_1 induces a semi-metric on Y , by relativization and we, by abuse of notation, use the same d_1 for the induced semi-metric on $Y = X - \{a\}$.

The product topology on $Y \times N^*$ is semi-metrizable via e given by $e = \left(d_1^2 + d_2^2 \right)^{1/2}$. Moreover, define

$$\begin{aligned} e(\pi, z) &= e(z, \pi) = e((y, \alpha), \pi) \\ &= d_1(y, a) \text{ for all } \alpha \in N \\ &= 1 \text{ for } \alpha = \omega \end{aligned}$$

$$e(\pi, \pi) = 0.$$

$$\begin{aligned} \text{Now } B_e(n, \pi) &= \{z \mid e(\pi, z) < 1/n \text{ and } z \in A\} \\ &= \{(y, \alpha) \mid \alpha \in N \text{ and } y \in Y \text{ with } d_1(y, a) < 1/n\} \cup \{\pi\} \\ &= [(B_1(a, n) - \{a\}) \times N] \cup \{\pi\}. \end{aligned}$$

Thus we see that the space (A, ρ) is semi-metrizable via e . That (A, ρ) is Hausdorff is easily verified.

Now let us define a new topology ρ' on $A = (Y \times N^*) \cup \{\pi\}$ as follows. Let $Y \times N^*$ have the product topology and the basic ρ' -open nbds of π be of the form $(0 \times N^*) \cup \{\pi\}$ where 0 is an open deleted nbd of a . We can define a semi-metric e' on $(Y \times N^*) \cup \{\pi\}$ such that e' is the same as e on $Y \times N^*$, while $e'(\pi, z) = e'(z, \pi) = e'((y, \alpha), \pi) = d_1(y, a)$ for all $\alpha \in N^*$. It is straightforward to verify that (A, ρ) and (A, ρ') are feebly compact.

The remainder of the proof consists of obtaining a violation of continuity similar to that in the proof of Theorem 2.1 of Raghavan and Reilly¹¹ or in Theorem 5.3 of Liu⁹.

The second part of the theorem is similar.

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ON s -CLOSED SPACES

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The authors define a topological space X to be s -closed if every cover of X by semi-open sets of X admits a finite subfamily whose semi-closures covers X . It is shown that cd -compactness due to Carnahan², weak RS -compactness due to Hong⁸ and s -closedness are all equivalent.

1. INTRODUCTION

A subset S of a topological space X is said to be semi-open⁹ if there exists an open set U such that $U \subset S \subset \text{Cl}(U)$. Thompson¹⁵ defined a topological space X to be S -closed if every cover of X by semi-open sets of X admits a finite subfamily whose closures covers X . The notion of the semi-closure was introduced by Crossley and Hildebrand³. In this paper, we introduce a class of topological spaces called s -closed spaces by utilizing the semi-closure. It will turn out that cd -compactness due to Carnahan², weak RS -compactness due to Hong⁸ and s -closedness are all equivalent.

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces on which no separation axiom is assumed unless explicitly stated. A subset S of a topological space is said to be 'regular open' (resp. 'regular closed') if $\text{Int}(\text{Cl}(S)) = S$ (resp. $\text{Cl}(\text{Int}(S)) = S$), where $\text{Cl}(S)$ (resp. $\text{Int}(S)$) denotes the closure (resp. interior) of S .

2. SEMI-REGULAR SETS

Let S be a subset of a space (X, τ) . The complement of a semi-open set is called semi-closed³. It is obvious that S is semiclosed if and only if $\text{Int}(\text{Cl}(S)) \subset S$. The intersection of all semi-closed sets containing S is called the semi-closure³ of S

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and is denoted by $s\text{Cl}(S)$. The semi-interior of S , denoted by $s\text{Int}(S)$, is defined by the union of all semi-open sets contained in S .

Definition 2.1—A subset S of a space (X, τ) is said to be semi-regular if it is both semi-open and semi-closed.

The family of all semi-open (resp. semi-regular) sets of X is denoted by $SO(X)$ (resp. $SR(X)$). For each $x \in X$, the family of all semi-open (resp. semi-regular) sets of X containing x is denoted by $SO(x)$ (resp. $SR(x)$). Cameron¹ defined a subset S of a space X to be regular semi-open if there exists a regular open set U of X such that $U \subset S \subset \text{Cl}(U)$. On the other hand, the first author⁵ defined a subset S of X to be semi regular open if $S = s\text{Int}(s\text{Cl}(S))$. However, these three notions are equivalent as shown by the following proposition.

Proposition 2.1—For a subset A of a space X , the following are equivalent :

- (a) $A \in SR(X)$,
- (b) $A = s\text{Int}(s\text{Cl}(A))$,
- (c) there exists a regular open set U of X such that $U \subset A \subset \text{Cl}(U)$.

PROOF : (a) \rightarrow (b) : If $A \in SR(X)$, then $s\text{Int}(s\text{Cl}(A)) = s\text{Int}(A) = A$. (b) \rightarrow (c): Suppose that $A = s\text{Int}(s\text{Cl}(A))$. Since $\text{Int}(\text{Cl}(S)) \subset s\text{Cl}(S)$ for every subset S of X (Noiri³, Lemma 4.14), $\text{Int}(\text{Cl}(A)) \subset s\text{Int}(s\text{Cl}(A)) = A$. Since $A \in SO(X)$, we have $A \subset \text{Cl}(\text{Int}(A))$. Therefore, we obtain

$$\text{Int}(\text{Cl}(A)) \subset A \subset \text{Cl}(\text{Int}(A)) \subset \text{Cl}(\text{Int}(\text{Cl}(A)))$$

where $\text{Int}(\text{Cl}(A))$ is regular open since $\text{Int}(\text{Cl}(\text{Int}(\text{Cl}(A)))) = \text{Int}(\text{Cl}(A))$.

(c) \rightarrow (a) : It is obvious that $A \in SO(X)$. We have $\text{Int}(\text{Cl}(A)) = \text{Int}(\text{Cl}(U)) = U \subset A$ and hence A is semi-closed. Thus, we obtain $A \in SR(X)$.

Proposition 2.2—If $A \in SO(X)$, then $s\text{Cl}(A) \in SR(X)$.

PROOF : Since $s\text{Cl}(A)$ is semi-closed, we show that $s\text{Cl}(A) \in SO(X)$. Since $A \in SO(X)$, $U \subset A \subset \text{Cl}(U)$ for open set U of X . Therefore, we have $U \subset s\text{Cl}(U) \subset s\text{Cl}(A) \subset \text{Cl}(U)$ and hence $s\text{Cl}(A) \in SO(X)$.

Remark 2.1 : According to the referee, Dorsett² in his Theorem 2.2 utilized the fact that $s\text{Cl}(A)$ is semi-open and semi-closed for each $A \in SO(X)$. However, he did not give the proof and Proposition 2.2 is remained.

A point $x \in X$ is said to be a semi θ -adherent point of a subset S of X if $s\text{Cl}(U) \cap S \neq \phi$ for every $U \in SO(x)$. The set of all semi θ -adherent points of S is called the semi θ -closure of S and is denoted by $s\text{Cl}_\theta(S)$. A subset S is called semi θ -closed if $s\text{Cl}_\theta(S) = S$.

Proposition 2.3—Let A be a subset of a space X . Then we have

(a) If $A \in SO(X)$, then $sCl(A) = sCl_\theta(A)$.

(b) If $A \in SR(X)$, then A is semi θ -closed.

PROOF : (a) In general, it holds that $sCl(A) \subset sCl_\theta(A)$. Assume that $x \notin sCl(A)$. Then, for some $U \in SO(x)$, $A \cap U = \phi$ and hence $A \cap sCl(U) = \phi$ since $A \in SO(X)$. This shows that $x \notin sCl_\theta(A)$. Therefore, $sCl(A) = sCl_\theta(A)$.

(b) This is obvious from (a).

Lemma 2.1—If O is open in a space X , then $sCl(O) = Int(Cl(O))$.

PROOF : For every subset S of X , $Int(Cl(S)) \subset sCl(S)$ (Noiri¹³, Lemma 4.14). We show the opposite inclusion. Assume that $x \notin Int(Cl(O))$. Then $x \in Cl(Int(X - O)) \in SO(X)$. Since O is open, we have $O \subset Int(Cl(O))$ and $O \cap Cl(Int(X - O)) = \phi$. This shows that $x \notin sCl(O)$. Therefore, we obtain $sCl(O) = Int(Cl(O))$.

A space (X, τ) is said to be extremally disconnected if $Cl(U) \in \tau$ for every $U \in \tau$.

Proposition 2.4—A space (X, τ) is extremally disconnected if and only if $Cl(U) = sCl(U)$ for every $U \in SO(X)$.

PROOF : *Necessity.* This follows from Lemma 1.13 of Noiri¹².

Sufficiency. For every $U \in \tau$, $U \in SO(X)$ and by Lemma 2.1 $Cl(U) = sCl(U) = Int(Cl(U))$. This shows that $Cl(U)$ is open for every $U \in \tau$.

3. s -CLOSED SPACES

Definition 3.1—A space (X, τ) is said to be s -closed if for every cover $\{V_\alpha \mid \alpha \in \nabla\}$ of X by semi-open sets of X , there exists a finite subset ∇_0 of ∇ such that $X = \cup \{sCl(V_\alpha) \mid \alpha \in \nabla_0\}$.

In Definition 3.1 the space (X, τ) is called S -closed¹⁵ if $sCl(V_\alpha)$ is replaced by $Cl(V_\alpha)$. By Proposition 2.4, s -closedness is coincident with S -closedness in an extremally disconnected space. A filter base \mathcal{F} on X is said to SR -converge to $x \in X$ if for each $V \in SR(x)$ there exists $F \in \mathcal{F}$ such that $F \subset V$. A filter base \mathcal{F} is said to SR -accumulate at $x \in X$ if $V \cap F \neq \phi$ for every $V \in SR(x)$ and every $F \in \mathcal{F}$.

Proposition 3.1—For a space (X, τ) , the following are equivalent :

- (a) X is s -closed,
- (b) every cover of X by semi-regular sets has a finite subcover,
- (c) for every family $\{V_\alpha \mid \alpha \in \nabla\}$ of semi-regular sets such that $\cap \{V_\alpha \mid \alpha \in \nabla\} = \phi$, there exists a finite subset ∇_0 of ∇ such that $\cap \{V_\alpha \mid \alpha \in \nabla_0\} = \phi$,
- (d) every filter base SR -accumulates at some point of X ,

(e) every maximal filter base SR -converges to some point of X .

PROOF : (a) \rightarrow (b) : This is obvious.

(b) \rightarrow (e) : Let \mathcal{F} be a maximal filter base on X . Assume that \mathcal{F} does not SR -converge to any point of X . Then \mathcal{F} does not SR -accumulate at any point of X . For each $x \in X$, there exists $F_x \in \mathcal{F}$ and $V_x \in SR(x)$ such that $V_x \cap F_x = \phi$. The family $\{V_x \mid x \in X\}$ is a cover of X by semi-regular sets of X . By (b), there exists a finite number of points x_1, x_2, \dots, x_n such that $X = \bigcup \{V_{x_i} \mid i = 1, 2, \dots, n\}$. Since \mathcal{F} is a filter base on X , there exists $F_0 \in \mathcal{F}$ such that $F_0 \subset \bigcap \{F_{x_i} \mid i = 1, 2, \dots, n\}$. Therefore, we have $F_0 = \phi$. This is a contradiction.

(e) \rightarrow (d) : Let \mathcal{F} be a filter base on X and \mathcal{F}_0 a maximal filter base such that $\mathcal{F} \subset \mathcal{F}_0$. By (e), \mathcal{F}_0 SR -converges to some $x \in X$. For every $F \in \mathcal{F}$ and every $V \in SR(x)$, there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subset V$. Therefore, we obtain $V \cap F \supset F_0 \cap F \neq \phi$. This shows that \mathcal{F} SR -accumulates at x .

(d) \rightarrow (c) : Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a family of semi-regular sets such that $\bigcap \{V_\alpha \mid \alpha \in \nabla\} = \phi$. Let $\Gamma(\nabla)$ denote the family of all finite subsets of ∇ . Assume that $\bigcap \{V_\alpha \mid \alpha \in \gamma\} \neq \phi$ for every $\gamma \in \Gamma(\nabla)$. Then, the family

$$\mathcal{F} = \left\{ \bigcap_{\alpha \in \gamma} V_\alpha \mid \gamma \in \Gamma(\nabla) \right\}$$

is a filter base on X . By (d), \mathcal{F} SR -accumulates at some $x \in X$. Since $\{X - V_\alpha \mid \alpha \in \nabla\}$ is a cover of X , $x \in X - V_{\alpha_0}$ for some $\alpha_0 \in \nabla$. Therefore, we have $X - V_{\alpha_0} \in SR(x)$ and $V_{\alpha_0} \in \mathcal{F}$. This is a contradiction.

(c) \rightarrow (a) : Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a cover of X by semi-open sets of X . By Proposition 2.2, $\{sCl(V_\alpha) \mid \alpha \in \nabla\}$ is a semi-regular cover of X . Thus $\{X - sCl(V_\alpha) \mid \alpha \in \nabla\}$ is a family of semi-regular sets of X having the empty intersection. By (c), there exists a finite subset ∇_0 of ∇ such that $\bigcap \{X - sCl(V_\alpha) \mid \alpha \in \nabla_0\} = \phi$; hence $X = \bigcup \{sCl(V_\alpha) \mid \alpha \in \nabla_0\}$. This shows that X is s -closed.

Corollary 3.1—For a space X , the following are equivalent :

(a) X is s -closed,

(b) every semi-open and semi-closed cover of X has a finite subcover (X is cd -compact due to Carnahan²),

(c) every regular semi-open cover of X has a finite subcover (X is weakly RS -compact due to Hong⁸).

PROOF : This is an immediate consequence of Proposition 2.1 and 3.1.

A space (X, τ) is said to be s -regular¹⁰ (resp. semi-regular⁷) if for each closed (resp. semi-closed) set F and each point $x \notin F$, there exist disjoint semi-open

sets U and V such that $x \in U$ and $F \subset V$. A space (X, τ) is said to be s -compact² if every semi-open cover of X has a finite subcover. It follows from Carnahan² that s -compactness implies s -closedness but not conversely and s -closedness neither implies compactness nor is implied by compactness.

Proposition 3.2—If (X, τ) is a semi-regular (resp. s -regular) s -closed space, then it is s -compact (resp. compact).

PROOF : Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a semi-open (resp. open) cover of X . For each $x \in X$, there exists $\alpha(x) \in \nabla$ such that $x \in V_{\alpha(x)}$. Since X is semi-regular (resp. s -regular), there exists $W_x \in SO(x)$ such that $sCl(W_x) \subset V_{\alpha(x)}$ (Dorsett⁷, Theorem 2.1) [resp. (Maheshwari and Prasad¹⁰, Theorem 2)]. The family $\{W_x \mid x \in X\}$ is a semi-open cover of X . By s -closedness of X , there exists a finite number of points x_1, x_2, \dots, x_n of X such that

$$X = \bigcup_{i=1}^n sCl(W_{x_i}) \subset \bigcup_{i=1}^n V_{\alpha(x_i)}.$$

Therefore, X is s -compact (resp. compact).

4. SETS s -CLOSED RELATIVE TO A SPACE

Definition 4.1—A subset S of a space X is said to be s -closed relative to X if for every cover $\{V_\alpha \mid \alpha \in \nabla\}$ of S by semi-open sets of X , there exists a finite subset ∇_0 of ∇ such that $S \subset \bigcup \{sCl(V_\alpha) \mid \alpha \in \nabla_0\}$.

Proposition 4.1—For a subset A of a space X , the following are equivalent :

- (a) A is s -closed relative to X ,
- (b) every cover of A by semi-regular sets of X has a finite subcover,
- (c) for every family $\{V_\alpha \mid \alpha \in \nabla\}$ of semi-regular sets of X such that $[\bigcap \{V_\alpha \mid \alpha \in \nabla\}] \cap A = \phi$, there exists a finite subset ∇_0 of ∇ such that $[\bigcap \{V_\alpha \mid \alpha \in \nabla_0\}] \cap A = \phi$,
- (d) every filter base on X which meets A SR -accumulates at some point of A ,
- (e) every maximal filter base on X which meets A SR -converges to some point of A .

PROOF : The proof is similar to that of Proposition 3.1 and is thus omitted.

Proposition 4.2—If K is a semi θ -closed set of an s -closed space (X, τ) , then K is s -closed relative to X .

PROOF : Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a cover of K by semi-regular sets of X . For each $x \in X - K$, there exists $U(x) \in SO(x)$ such that $sCl(U(x)) \subset X - K$. By Proposition 2.2, $sCl(U(x)) \in SR(x)$ for each $x \in X$ and the family

$$\{sCl(U(x)) \mid x \in X - K\} \cup \{V_\alpha \mid \alpha \in \nabla\}$$

is a cover of X by semi-regular sets of X . Since X is s -closed, there exists a finite subset ∇_0 of ∇ such that $K \subset \bigcup \{V_\alpha \mid \alpha \in \nabla_0\}$. Thus, by Proposition 4.1 K is s -closed relative to X .

A space (X, τ) is said to be weakly-Hausdorff (simply weakly- T_2^1)¹⁴ if every point of X is the intersection of regular closed sets of X .

Proposition 4.3—Let (X, τ) be a weakly- T_2 space. If K is s -closed relative to X , then K is semi θ -closed.

PROOF : Let $x \in X - K$. For each $k \in K$, there exists a regular closed set F_k such that $k \in F_k$ and $x \notin F_k$. Since $F_k \in SR(X)$ and $K \subset \bigcup \{F_k \mid k \in K\}$, by Proposition 4.1 we have $K \subset \bigcup \{F_k \mid k \in K_0\}$ for some finite subset K_0 of K . Now, put $V = \bigcap \{X - F_k \mid k \in K_0\}$. Then $V \in SR(x)$ and $V \cap K = \phi$. Therefore, $x \notin sCl_\theta(K)$ and hence K is semi θ -closed.

5. THE QUASI-IRRESOLUTE IMAGES OF s -CLOSED SPACES

Definition 5.1—A function $f: X \rightarrow Y$ is said to be quasi-irresolute⁶ if for each $x \in X$ and each $V \in SO(f(x))$ there exists $U \in SO(x)$ such that $f(U) \subset sCl(V)$.

A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be irresolute⁴ (resp. semi-continuous⁹) if $f^{-1}(V) \in SO(X)$ for every $V \in SO(Y)$ (resp. $V \in \sigma$).

Lemma 5.1 (Di Maio and Noiri⁶)—Let $f: X \rightarrow Y$ be a function. Then the following statements hold :

- (a) quasi-irresoluteness and continuity are independent of each other,
- (b) irresoluteness implies both quasi-irresoluteness and semi-continuity which are independent of each other,
- (c) f is quasi-irresolute if and only if $f^{-1}(V) \in SR(X)$ for every $V \in SR(Y)$.

Corollary 5.1—If $f: X \rightarrow Y$ is a quasi-irresolute function and K is s -closed relative to X , then $f(K)$ is closed relative to Y .

PROOF : Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a cover of $f(K)$ by semi-regular sets of Y . Since f is quasi-irresolute, by Lemma 5.1 $\{f^{-1}(V_\alpha) \mid \alpha \in \nabla\}$ is a cover of K by semi-regular sets of X . By Proposition 4.1, there exists a finite subset ∇_0 of ∇ such that $K \subset \bigcup \{f^{-1}(V_\alpha) \mid \alpha \in \nabla_0\}$. Therefore, we have $f(K) \subset \bigcup \{V_\alpha \mid \alpha \in \nabla_0\}$. It follows from Proposition 4.1 that $f(K)$ is s -closed relative to Y .

Proposition 5.1—(a) s -closedness is preserved by open continuous surjections.

(b) s -closedness is projective but not productive.

PROOF : (a) Every open continuous function is irresolute (Crossley and Hildebrand⁴, Theorem 18) and, Lemma 5.1, every irresolute function is quasi-irresolute. Therefore, this follows immediately from Corollary 5.1.

(b) Since the natural projection is open continuous, then s -closedness is projective. It is shown in Thompson¹⁵ that $\beta\mathbb{N}$ is extremally disconnected and S -closed. Therefore, $\beta\mathbb{N}$ is s -closed but $\beta\mathbb{N} \times \beta\mathbb{N}$ is not S -closed and hence it is not s -closed.

A function $f: X \rightarrow Y$ is said to be semi θ -closed if $f(F)$ is semi θ -closed in Y for every semi θ -closed set F of X .

Corollary 5.2—If $f: X \rightarrow Y$ is a quasi-irresolute function, X is s -closed and Y is weakly- T_2 , then f is semi θ -closed.

PROOF: Let F be a semi θ -closed set of X . By Proposition 4.2, F is s -closed relative to X and $f(F)$ is s -closed relative to Y by Corollary 5.1. Since Y is weakly- T_2 , by Proposition 4.3 $f(F)$ is semi θ -closed and hence f is semi θ -closed.

A space (X, τ) is said to be semi- T_2 (Maheshwari and Prasad¹¹) if for each pair x, y of distinct points of X , there exists disjoint semi-open sets U and V such that $x \in U$ and $y \in V$.

Since a regular closed set is semi-regular, weakly- T_2 spaces are semi- T_2 . However, the converse is not true as the following example shows.

Example 5.1—Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then a space (X, τ) is semi- T_2 but not weakly- T_2 .

By Corollary 5.1, it is obvious that the quasi-irresolute image of an s -closed space in any weakly- T_2 space is semi θ -closed. Moreover, we have the following proposition.

Proposition 5.2—The quasi-irresolute image of an s -closed space in any semi- T_2 space is semi θ -closed.

PROOF: Let $f: X \rightarrow Y$ be quasi-irresolute, X s -closed and Y semi- T_2 . Assume that $y \in sCl_\theta(f(X))$. Let $[SO(y)]$ be a filter generated by the family $SO(y)$. Since X is s -closed, by Proposition 3.1 the filter base $f^{-1}([SO(y)])$ has an SR -accumulation point x in X . Now, let V be any semi-regular set containing $f(x)$. By Lemma 5.1, we have $f^{-1}(V) \in SR(x)$. Therefore, for every $W \in SO(y)$, $f^{-1}(V) \cap f^{-1}(W) \neq \phi$ and hence $V \cap W \neq \phi$. Since Y is semi- T_2 , by using Proposition 2.2 we obtain $f(x) = y$ and hence $y \in f(X)$. This shows that $f(X)$ is semi θ -closed.

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NEW SUBSPACES OF EXTENSIONS, CONNECTEDNESS AND LOCAL CONNECTEDNESS

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In this paper, we introduce some subspaces of extensions, where a space is said to be an extension of a space X if it contains X as a dense subspace. That is, for each infinite cardinal κ , a subspace $E(Y, X, \kappa)$ of Y which contains X is defined. It is shown that $E(\beta X, X, \omega) = \beta X$, $E(\beta X, X, \omega_1) = \nu X$ and $\bigcap_{\kappa} E(\beta X, X, \kappa) = \mu X$ for each completely regular space X . Furthermore it is shown that X is connected if and only if $E(Y, X, \omega)$ is so, and that X is locally connected if and only if $E(Y, X, \omega)$ is so at each point of $\bigcap_{\kappa} E(Y, X, \kappa)$ and not at each point of the outside of it, and that X is pseudocompact if and only if $E(Y, X, \omega) = \bigcap_{\kappa} E(Y, X, \kappa)$ for each extension Y of X .

INTRODUCTION

In this paper, all spaces are assumed to be regular. It is well known that X is connected if and only if βX is connected, and that X is locally connected if and only if βX is locally connected at each point of μX but not at each point of $\beta X - \mu X$. In this paper, we will generalize the above results for every extension by defining new subspaces of extensions of X . Here, a space Y is said to be an extension of X if Y contains X as a dense subspace, furthermore βX denotes the Stone-Čech compactification of X , μX the completion with respect to the universal uniformity of X (i. e. the uniformity of all continuous pseudometrics on $X \times X$), respectively.

In section 1, we will define subspaces of extensions of a space and consider the relations between such subspaces, the Stone-Čech compactification and pseudocompactness.

In section 2, we will consider connectedness and local connectedness of some extensions.

1. SUBSPACES OF EXTENSIONS AND THE STONE-ČECH COMPACTIFICATIONS

In this section, we will define subspaces of extensions, and consider the relations between such subspaces, the Stone-Čech compactifications and pseudocompactness.

Definitions 1.1—Let Y be an extension of X . For a subset A of X , $O_Y(A)$ denotes the set $Y - \text{cl}_Y(X - A)$. If there is no confusion in the context, we will write $O(A)$ instead of $O_Y(A)$. For an infinite cardinal number κ , a subset A of Y is called κ -connected set of X in Y if for each cellular family $\{U_\alpha\}_{\alpha \in \Lambda}$ of open sets of X (i. e. $U_\alpha \cap U_\beta = \emptyset$ for each distinct α and β of Λ) with $|\Lambda| < \kappa$, if $A \subset O(\bigcup_{\alpha \in \Lambda} U_\alpha)$ holds, then there is an $\alpha \in \Lambda$ such that A is contained in $O(U_\alpha)$. Particularly, if for a point p of Y , $\{p\}$ is a κ -connected set, we say that p is κ -connected point of X in Y (cf. Banaschewski^{1,2}). The κ -extension $E(Y, X, \kappa)$ of X in Y consists of all points of Y which are κ -connected points of X in Y . Furthermore the complete extension $E(Y, X)$ of X in Y consists of all points of Y which are κ -connected points of X in Y for every infinite cardinal κ . Clearly $E(Y, X) = \bigcap \{E(Y, X, \kappa) : \kappa \text{ is an infinite cardinal}\}$. Finally, Y is said to be κ -extension (complete extension) of X if $Y = E(Y, X, \kappa)$ ($Y = E(Y, X)$, respectively). Then evidently for each infinite cardinals κ and λ with $\kappa \leq \lambda$, $X \subset E(Y, X) \subset E(Y, X, \lambda) \subset E(Y, X, \kappa) \subset Y$ holds.

Lemma 1.2—Let Y be an extension of X and κ an infinite cardinal. Suppose $\{U_\alpha\}_{\alpha \in A}$ is a cellular family of open sets of X .

Then the following hold;

$$(1) \quad O\left(\bigcup_{\alpha \in A} U_\alpha\right) \cap E(Y, X, \kappa) = \bigcup_{\alpha \in A} [O(U_\alpha) \cap E(Y, X, \kappa)]$$

if the cardinality of A is less than κ ,

$$(2) \quad O\left(\bigcup_{\alpha \in A} U_\alpha\right) \cap E(Y, X) = \bigcup_{\alpha \in A} [O(U_\alpha) \cap E(Y, X)].$$

The proof of the above lemma is straightforward.

Lemma 1.3—Let Y be an extension of X , $\{U_\alpha\}_{\alpha \in A}$ a cellular family of open sets of X with $p \in O(\bigcup_{\alpha \in A} U_\alpha)$. Suppose there are a zero set Z and a cozero set V of Y such that -

$$p \in Z \subset V \subset O\left(\bigcup_{\alpha \in A} U_\alpha\right).$$

Furthermore define for each $\alpha \in A$, $Z_\alpha = (Z \cap X) \cap U_\alpha$ and $V_\alpha = (V \cap X) \cap U_\alpha$. Then for each subset B of A , $\bigcup_{\alpha \in B} V_\alpha$ is a cozero set of X and $\bigcup_{\alpha \in B} Z_\alpha$ is a zero set of X .

Consequently, $Z_\alpha \subset \subset U_\alpha$ (i. e. Z_α is completely separated from $X - U_\alpha$) for each $\alpha \in A$, and the family $\{Z_\alpha\}_{\alpha \in A}$ is discrete in X .

PROOF : Suppose that the condition of this lemma is satisfied. Then since V is a cozero set of Y , there is a continuous function f from X to the unit interval I such that $V \cap X = C_z(f)$, where $C_z(f)$ denotes all points of X whose values by f are non-zero. Let B be a subset of A . Define a function f_B from X to I which satisfies the following;

$$f_B(x) = \begin{cases} f(x) & \text{if } x \in \bigcup_{\alpha \in B} V_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then evidently f_B is continuous, furthermore $Cz(f_B) = \bigcup_{\alpha \in B} V_\alpha$.

Hence $\bigcup_{\alpha \in B} V_\alpha$ is a cozero set of X .

Next let $Z \cap X = Z(g)$ for some continuous function g from X to I . Where $Z(g)$ denotes all points of X whose values by g are zero. Let $g_B = g + f_A - f_B$. Then evidently g_B is continuous and $\bigcup_{\alpha \in B} Z_\alpha = Z(g_B)$. Hence $\bigcup_{\alpha \in B} Z_\alpha$ is a zero set of X .

Theorem 1.4—Let Y be a completely regular extension of X and p a point of $Y - E(Y, X, \omega)$. Then there exists a function of $C^*(X)$ which is not continuously extendable over $X \cup \{p\}$ (i. e. as an element of $C^*(X \cup \{p\})$).

PROOF : Suppose a point p of Y is not in $E(Y, X, \omega)$. Then there is a cellular family $\{U_i\}_{i=1}^n$ of open sets of X such that $p \in O(\bigcup_{i=1}^n U_i)$ and $p \notin O(U_i)$ for $i = 1, \dots, n$. Let Z be a zero set neighbourhood of p in Y which is completely separated from $Y - O(\bigcup_{i=1}^n U_i)$. Let Z_i be the set $(Z \cap X) \cap U_i$ for each $i = 1, \dots, n$. By Lemma 1.3, for each $i = 1, \dots, n$ there exists a continuous function f_i from X to I such that

$$f_i(x) = \begin{cases} 0 & \text{if } x \notin U_i \\ 1 & \text{if } x \in Z_i. \end{cases}$$

Then $f = \sum_{i=1}^n i \cdot f_i$ (i. e. $f(x) = 1 \cdot f_1(x) + 2 \cdot f_2(x) + \dots + n \cdot f_n(x)$ for each x of X) is a continuous bounded real valued function on X . Evidently this function can be continuously extendable over $X \cup \{p\}$.

Corollary 1.5— $E(\beta X, X, \omega) = \beta X$ for each completely regular space X .

Theorem 1.6—Let Y be a completely regular extension of X and p a point of $Y - E(Y, X, \omega_1)$. Then there exists a function of $C(X)$ which is not continuously extendable over $X \cup \{p\}$ (i. e. as an element of $C(X \cup \{p\})$).

PROOF : Let p be a point of $Y - E(Y, X, \omega_1)$. Then there exists a cellular family $\{U_i\}_{i \in \omega}$ of open sets of X such that $p \in O(\bigcup_{i \in \omega} U_i)$ and $p \notin O(U_i)$ for each $i \in \omega$. Let Z be a zero set neighbourhood of p in Y which is completely separated from $Y - O(\bigcup_{i \in \omega} U_i)$. Let Z_i be the set $(Z \cap X) \cap U_i$ for each $i \in \omega$. According to 1.3, for each $i \in \omega$, there exists a continuous function f_i from X to I such that

$$f_i(x) = \begin{cases} 0 & \text{if } x \notin U_i \\ 1 & \text{if } x \in U_i. \end{cases}$$

Then $f = \sum_{i \in \omega} i \cdot f_i$ is a continuous function from X to the real line. Evidently this function can not be continuously extendable over $X \cup \{p\}$.

Theorem 1.7— $E(\beta X, X, \omega_1) = \upsilon X$ for each completely regular space X , where υX denotes the Hewitt realcompactification of X .

PROOF: By Theorem 1.6, $F(\beta X, X, \omega_1)$ contains υX .

Next, suppose p is not in υX . Then there is a continuous function f from βX to I such that $p \in Z(f)$ and $Z(f) \cap X = \emptyset$. Then by the usual way, one can construct a cellular family $\{U_i\}_{i \in \omega}$ of open sets of X such that $p \in O(\bigcup_{i \in \omega} U_i)$ and $p \notin O(U_i)$ for each $i \in \omega$ by use of the function f . Hence p is not in $E(\beta X, X, \omega_1)$.

Theorem 1.8—Let κ be an uncountable non-measurable cardinal. Then $E(\beta X, X, \kappa) = \upsilon X$ for each completely regular space X .

PROOF: By Theorem 1.7, evidently $E(\beta X, X, \kappa) \subset \upsilon X$ holds.

Next suppose p is a point of υX and $\{U_\alpha\}_{\alpha \in A}$ is a cellular family of open sets of X with $|A| < \kappa$ such that $p \in O(\bigcup_{\alpha \in A} U_\alpha)$. Let Z be a zero set neighbourhood of p in βX which is completely separated from $\beta X - O(\bigcup_{\alpha \in A} U_\alpha)$. Let Z_α be the set $(Z \cap X) \cap U_\alpha$ for each $\alpha \in A$. Then by 1.3, for each subset B of A , $\bigcup_{\alpha \in B} Z_\alpha$ is a zero set of X . Furthermore $Z \cap X = \bigcup_{\alpha \in A} Z_\alpha$ is in \mathcal{H}^p , where \mathcal{H}^p denotes all zero set of X whose closures in βX contain the point p . Then there exists an $\alpha \in A$ such that $Z_\alpha \in \mathcal{H}^p$, because if for each $\alpha \in A$, Z_α is not in \mathcal{H}^p , then we can define a free ultrafilter \mathcal{F} on A by $B \in \mathcal{F}$ iff $\bigcup_{\alpha \in B} Z_\alpha \in \mathcal{H}^p$. Since \mathcal{H}^p is a real z -ultrafilter, \mathcal{F} has the countable intersection property. Then \mathcal{F} defines a countably additive non-trivial $\{0, 1\}$ -valued measure on A . But this contradicts to the fact that κ is non-measurable and the cardinality of A is less than κ . Hence there is an $\alpha \in A$ such that $Z_\alpha \in \mathcal{H}^p$. Therefore $p \in \text{cl}_{\beta X} Z_\alpha$. Since by Lemma 1.3 $\text{cl}_{\beta X} Z_\alpha \subset O(U_\alpha)$, p is in $O(U_\alpha)$.

In a similar way one can prove the next results.

Corollary 1.9—Let γ be the first measurable cardinal if exists. Then $E(\beta X, X, \gamma) = \upsilon X$ for each completely regular space X .

Theorem 1.10—Let Y be a completely regular extension of X and p a point of $Y - E(Y, X)$. Then there exists a continuous pseudometric on X which is not continuously extendable over $X \cup \{p\}$ as a pseudometric.

PROOF : Assume that p is a point of $Y - E(Y, X)$. Then there exists a cellular family $\{U_\alpha\}_{\alpha \in A}$ of open sets of X such that $p \in O(\bigcup_{\alpha \in A} U_\alpha)$ and $p \notin O(U_\alpha)$ for each $\alpha \in A$. Let Z be a zero set neighbourhood of p in Y which is completely separated from $Y - O(\bigcup_{\alpha \in A} U_\alpha)$. Define Z_α to be the set $(Z \cap X) \cap U_\alpha$ for each $\alpha \in A$. Since $Z_\alpha \subset \subset U_\alpha$, there exists a continuous function f_α from X to I such that

$$f_\alpha(x) = \begin{cases} 0 & \text{if } x \notin U_\alpha \\ 1 & \text{if } x \in Z_\alpha \end{cases}$$

for each $\alpha \in A$. Define $d(x, y) = \sum_{\alpha \in A} |f_\alpha(x) - f_\alpha(y)|$. Since for each $x \in X$, $f_\alpha(x)$ is non-zero at most one α , d is well defined. Evidently d is a continuous pseudometric on X which can not be continuously extendable over $X \cup \{p\}$ as a pseudometric.

Theorem 1.11— $E(\beta X, X) = \mu X$ for each completely regular space X .

PROOF : By Theorem 1.10, $\mu X \subset E(\beta X, X)$.

Next let p be a point of $E(\beta X, X)$. Assume p is not in μX . Then by Theorem 1.7, p is in νX . Hence \mathcal{A}^p is a real z -ultrafilter and not a Cauchy z -filter. Hence there is a continuous pseudometric d on X and an $\epsilon > 0$ such that $d(Z) \geq \epsilon$ for each $Z \in \mathcal{A}^p$. But by (15.18) of Gillman and Jerison⁵, there exists a family $\{Z_\alpha\}_{\alpha \in A}$ of zero sets of X such that the following properties hold;

- (1) $\bigcup_{\alpha \in A} Z_\alpha \in \mathcal{A}^p$,
- (2) $d(Z_\alpha) < \epsilon/4$,
- (3) $\{Z_\alpha\}_{\alpha \in A}$ is d -discrete,
- (4) for each subset B of A , $\bigcup_{\alpha \in B} Z_\alpha$ is a zero set of X .

Let δ be the gauge of $\{Z_\alpha\}_{\alpha \in A}$ by d , define

$$U_\alpha = \{y \in X : d(y, Z_\alpha) < \min(\epsilon/4, \delta/2)\},$$

$$Z'_\alpha = \{y \in X : d(y, Z_\alpha) \leq \min(\epsilon/4, \delta/2)\}.$$

Then for each $\alpha \in A$, U_α is an open set of X and Z_α is a zero set of X . Furthermore, $\{U_\alpha\}_{\alpha \in A}$ is a cellular family of open sets of X . Since $\bigcup_{\alpha \in A} Z_\alpha \subset \subset \bigcup_{\alpha \in A} U_\alpha$ and $\bigcup_{\alpha \in A} Z_\alpha \in \mathcal{A}^p$, p is a point of $O(\bigcup_{\alpha \in A} U_\alpha)$. But since $d(Z'_\alpha) (\leq 3\epsilon/4 < \epsilon, Z'_\alpha \notin \mathcal{A}^p$. Furthermore since $U_\alpha \subset Z'_\alpha$, p is not in $O(U_\alpha)$ for each $\alpha \in A$. This contradicts the fact that p is in $E(\beta X, X)$.

Theorem 1.12—Let γX and αX be compactifications of a completely regular space X and p a point of αX . If there exists a continuous function f from γX to αX which leaves X pointwise fixed, then p is in $E(\alpha X, X, \kappa)$ if and only if $f^{-1}(p)$ is κ -connected for each infinite cardinal κ .

PROOF : Assume p is in $E(\alpha X, X, \kappa)$. Let $\{U_\alpha\}_{\alpha \in A}$ be a cellular family of open sets of X with $|A| < \kappa$ such that $f^{-1}(p)$ is contained in $O_{\gamma X}(\bigcup_{\alpha \in A} U_\alpha)$. Since $f^{-1}(p)$ is disjoint from $\text{cl}_{\gamma X}(X - \bigcup_{\alpha \in A} U_\alpha)$ and f is perfect, p is not in $\text{cl}_{\alpha X}(X - \bigcup_{\alpha \in A} U_\alpha)$. Hence p is in $O_{\alpha X}(\bigcup_{\alpha \in A} U_\alpha)$. But p is in $E(\alpha X, X, \kappa)$, therefore there is an $\alpha \in A$ such that p is in $O_{\alpha X}(U_\alpha)$, that is, p is not in $\text{cl}_{\alpha X}(X - U_\alpha)$. Hence $f^{-1}(p)$ is disjoint from $f^{-1}(\text{cl}_{\alpha X}(X - U_\alpha))$. Furthermore since f is continuous, $f^{-1}(p)$ is disjoint from $\text{cl}_{\gamma X}(X - U_\alpha)$. Therefore $f^{-1}(p)$ is contained in $O_{\gamma X}(U_\alpha)$. So $f^{-1}(p)$ is κ -connected. Next assume $f^{-1}(p)$ is κ -connected and $\{U_\alpha\}_{\alpha \in A}$ is cellular family of open sets of X with $|A| < \kappa$ such that p is in $O_{\alpha X}(\bigcup_{\alpha \in A} U_\alpha)$. Then since f is a function which leaves X pointwise fixed, $f^{-1}(p) \subset f^{-1}(O_{\alpha X}(\bigcup_{\alpha \in A} U_\alpha)) \in O_{\gamma X}(\bigcup_{\alpha \in A} U_\alpha)$ holds. Because $f^{-1}(p)$ is κ -connected, there is an $\alpha \in A$ such that $f^{-1}(p)$ is contained in $O_{\gamma X}(U_\alpha)$, that is, $f^{-1}(p) \cap \text{cl}_{\gamma X}(X - U_\alpha) = \phi$. Since f is perfect, p is not in $\text{cl}_{\alpha X}(X - U_\alpha)$, that is, p is in $O_{\alpha X}(U_\alpha)$. Hence p is κ -connected.

Lemma 1.13—Let Y be an ω -compactification (i. e. compactification and ω -extension) of a completely regular space X . If Z is non-empty closed set of Y , then Z is connected if and only if Z is ω -connected.

PROOF : Assume Z is a (closed) set of Y and $\{U_i\}_{i=1}^n$ is a cellular family of open sets of X with $Z \subset O(\bigcup_{i=1}^n U_i)$. If Z is connected, then clearly there is an $i = 1, \dots, n$ such that Z is contained in $O(U_i)$. Hence Z is ω -connected.

Next assume Z is not connected. Since Z is closed and Y is normal space, there are disjoint open sets V_1 and V_2 in Y such that Z is contained in $V_1 \cup V_2$ and $Z \cap V_i$ is non-empty for $i = 1, 2$. Define $U_i = V_i \cap X$ for $i = 1, 2$. Evidently $\{U_1, U_2\}$ satisfies $Z \subset O(U_1 \cup U_2)$ but $Z \not\subset O(U_i)$ for $i = 1, 2$. So Z is not ω -connected.

By Theorem 1.12 and Lemma 1.13, the next results hold.

Corollary 1.14—Let γX and αX be compactifications of a completely regular space X , and p a point of αX . If there is a continuous function f from γX to αX which leaves X pointwise fixed, furthermore γX is an ω -extension of X , then p is in $E(\alpha X, X, \omega)$ if and only if $f^{-1}(p)$ is connected.

Corollary 1.15—Let αX be a compactification of completely regular space X , and f a continuous function from βX to αX which leaves X pointwise fixed, then the following hold;

- (1) for each point p of αX , p is in $E(\alpha X, X, \omega)$ if and only if $f^{-1}(p)$ is connected.
- (2) αX is an ω -extension if and only if for each point p of αX , $f^{-1}(p)$ is connected.

In the rest of this section, we will show the relations of pseudocompactness and some extensions.

Theorem 1.16—Let γX and αX be compactifications of a completely regular space X . If there exists a continuous function f from γX to αX which leaves X pointwise fixed, then for each cardinal κ , $f^{-1}(E(\alpha X, X, \omega) - E(\alpha X, X, \kappa)) \subset \gamma X - E(\gamma X, X, \kappa)$ holds. Consequently, $f^{-1}(E(\alpha X, X, \omega) - E(\alpha X, X)) \subset \gamma X - E(\gamma X, X)$.

PROOF : Let p be a point of $E(\alpha X, X, \omega) - E(\alpha X, X, \kappa)$. Then there is a cellular family $\{U_\alpha\}_{\alpha \in A}$ of open sets of X with $|A| < \kappa$ such that p is in $O_{\alpha X}(\bigcup_{\alpha \in A} U_\alpha)$ but not in $O_{\alpha X}(U_\alpha)$ for each $\alpha \in A$. Then $f^{-1}(p) \subset f^{-1}(O_{\alpha X}(\bigcup_{\alpha \in A} U_\alpha)) \subset O_{\gamma X}(\bigcup_{\alpha \in A} U_\alpha)$ holds. Assume $f^{-1}(p)$ is not disjoint from $E(\gamma X, X, \kappa)$. Let x be a point of $f^{-1}(p) \cap E(\gamma X, X, \kappa)$. Since there is an $\beta \in A$ such that x is in $O_{\gamma X}(U_\beta)$, x is in $\text{cl}_{\gamma X} U_\beta$. Hence $p = f(x) \in f(\text{cl}_{\gamma X} U_\beta) = \text{cl}_{\alpha X} U_\beta$. Therefore p is not in $O_{\alpha X}(\bigcup_{\alpha \in A - \{\beta\}} U_\alpha)$. But by the assumption p is not in $O_{\alpha X}(U_\beta)$. This contradicts to the fact that p is in $E(\alpha X, X, \omega)$. Hence $f^{-1}(p)$ is disjoint from $E(\gamma X, X, \kappa)$. So $f^{-1}(E(\alpha X, X, \omega) - E(\alpha X, X, \kappa)) \subset \gamma X - E(\gamma X, X, \kappa)$ holds.

Corollary 1.17—Let αX be a compactification of a completely regular space X and f a continuous function from βX to αX which leaves X pointwise fixed. Then the following hold:

- (1) $f^{-1}(E(\alpha X, X, \omega) - E(\alpha X, X, \kappa)) \subset \beta X - E(\beta X, X, \kappa)$ for each infinite cardinal κ ,
- (2) $f^{-1}(E(\alpha X, X, \omega) - E(\alpha X, X)) \subset \beta X - \mu X$.

The following result is well known³, but we prove it using Theorem 1.11.

Lemma 1.18—Let X be a completely regular space. Then X is a pseudocompact space (i. e. $\beta X = \nu X$) if and only if $\beta X = \mu X$.

PROOF : Suppose X is pseudocompact and p is a point of $\beta X - \mu X$. Then there exists a cellular family $\{U_\alpha\}_{\alpha \in A}$ of open sets of X such that $p \in O(\bigcup_{\alpha \in A} U_\alpha)$ but $p \notin O(U_\alpha)$ for each $\alpha \in A$. Let Z be a zero set neighbourhood of p in βX which is completely separated from $\beta X - O(\bigcup_{\alpha \in A} U_\alpha)$. Define Z_α to be the set $(Z \cap X) \cap U_\alpha$ for each $\alpha \in A$. Since p is in νX , $\{\alpha \in A : Z_\alpha \text{ is non-empty}\}$ has (uncountably many) infinite members. Let $\{\alpha_i : i \in \omega\}$ be a countably infinite subset of it. Define for each $i \in \omega$ a continuous function f_i from X to I such that

$$f_i(x) = \begin{cases} 0 & \text{if } x \notin U_i \\ 1 & \text{if } x \in U_i. \end{cases}$$

Let define $f = \sum_{i \in \omega} i; f_i$. Then evidently f is unbounded and continuous. This contradicts the fact that X is pseudocompact. Since $\mu X \subset \nu X \subset \beta X$, the converse is true.

Corollary 1.19—Let X be a completely regular space. Then the following are equivalent :

- (1) X is pseudocompact,
- (2) for each compactification Y of X , $E(Y, X, \omega) = E(Y, X)$ holds,
- (3) for each uncountable cardinal κ and each compactification Y of X , $E(Y, X, \omega) = E(Y, X, \kappa)$ holds,
- (4) there exists an uncountable cardinal κ such that for each compactification Y of X , $E(Y, X, \omega) = E(Y, X, \kappa)$ holds.

PROOF : (1) \rightarrow (2); Let X be pseudocompact and f be a continuous function from βX to Y which leaves X pointwise fixed. Then by Corollary 1.17 and Lemma 1.18, (2) holds.

(2) \rightarrow (3) and (3) \rightarrow (4) are evident.

(4) \rightarrow (1); Let (4) hold. Since $E(\beta X, X, \kappa) \subset E(\beta X, X, \omega_1) \subset E(\beta X, X, \omega)$, $\beta X = \nu X$.

Remark 1.20 : As compared with the above corollary, naturally we have the next question :

If there exists a compactification Y of X such that $E(Y, X, \omega) = E(Y, X)$, then is X pseudocompact?

But we can find the negative answer. Let $R^* = R \cup \{p\}$ be the one-point compactification of the real line R . Then $E(R^*, R, \omega) = E(R^*, R) = R$, because let $U_0 = \{r \in R : r > 0\}$ and $U_1 = \{r \in R : r < 0\}$. Then p is in $O(U_0 \cup U_1)$ but not in $O(U_i)$ for $i = 0, 1$. But R is not pseudocompact.

Lemma 1.21—Let X, Y and Z be spaces such that X is dense in Y and Y is dense in Z , and κ be an infinite cardinal. Then $E(Z, X, \kappa) \subset E(Z, Y, \kappa)$ holds.

Consequently, $E(Z, X) \subset E(Z, Y)$ holds.

PROOF : Let p be a point of $E(Z, X, \kappa)$ and $\{U_\alpha\}_{\alpha \in A}$ a cellular family of open sets of Y with $|A| < \kappa$ such that p is in $O_Z(\bigcup_{\alpha \in A} U_\alpha)$. Since Z is regular, there exists an open set W in Z such that $p \in W \subset \text{cl}_Z W \subset O_Z(\bigcup_{\alpha \in A} U_\alpha)$ holds. Let W_α be the set $(W \cap X) \cap U_\alpha$ for each $\alpha \in A$. Note that W_α is open in X for each $\alpha \in A$. Then clearly p is in $O_Z(\bigcup_{\alpha \in A} W_\alpha)$. Since p is in $E(Z, X, \kappa)$, there is an $\alpha \in A$ such

that p is in $O_Z(W_\alpha)$. Note that $\text{cl}_Z W_\alpha \cap Y \subset U_\alpha$ for each $\alpha \in A$. Then $O_Z(W_\alpha) \cap Y \subset U_\alpha$, hence $p \in O_Z(W_\alpha) \subset O_Z(U_\alpha)$ for each $\alpha \in A$. Hence p is a point of $E(Z, Y, \kappa)$.

The proofs of the next lemmas are straightforward.

Lemma 1.22—Let Y be an extension of X and κ an infinite cardinal. Then $E(Y, E(Y, X, \kappa), \kappa) = E(Y, X, \kappa)$ holds.

Lemma 1.23—Let X, Y and Z be spaces such that X is dense in Y and Y is dense in Z , and κ be an infinite cardinal. Then the following hold;

- (1) if Y is contained in $E(Z, X, \kappa)$, then $E(Z, X, \kappa) = E(Z, Y, \kappa)$ holds,
- (2) if Y is contained in $E(Z, X)$, then $E(Z, X) = E(Z, Y)$ holds.

Lemma 1.24—Let X, Y and Z be spaces such that X is dense in Y and Y is dense in Z , and κ be an infinite cardinal. Then $E(Y, X, \kappa) = E(Z, X, \kappa) \cap Y$ and $E(Y, X) = E(Z, X) \cap Y$ hold.

Theorem 1.25—Let X be a completely regular space. Then the following are equivalent :

- (1) X is pseudocompact,
- (2) for each completely regular extension Y of X , $E(Y, X, \omega) = E(Y, X)$ holds,
- (3) for each uncountable cardinal κ and each completely regular extension Y of X , $E(Y, X, \omega) = E(Y, X, \kappa)$ holds.
- (4) there exists an uncountable cardinal κ such that for each completely regular extension Y of X , $E(Y, X, \omega) = E(Y, X, \kappa)$ holds.

PROOF : (1) \rightarrow (2); Assume X is pseudocompact. Let Z be a compactification of Y . Then by corollary 1.19, $E(Z, X, \omega) = E(Z, X)$. Hence $E(Y, X, \omega) = E(Z, X, \omega) \cap Y = E(Z, X) \cap Y = E(Y, X)$ by Lemma 1.24.

(2) \rightarrow (3) and (2) \rightarrow (4) are evident. Also (4) \rightarrow (1) is true by (4) \rightarrow (1) of Corollary 1.19.

2. CONNECTEDNESS AND LOCAL CONNECTEDNESS OF EXTENSIONS

It is well known that X is connected if and only if βX is connected (cf. Engelking⁴, 6.1.14). We can show the next.

Theorem 2.1—Let Y be an extension of X . Then the following are equivalent;

- (1) X is connected,
- (2) every subspace of $E(Y, X, \omega)$, which contains X , is connected,
- (3) there exists a connected subspace of $E(Y, X, \omega)$ which contains X ,
- (4) $E(Y, X, \omega)$ is connected.

PROOF : (1) \rightarrow (2) \rightarrow (3) \rightarrow (4) is evident.

To prove (4) \rightarrow (1), assume X is not connected. Then there are disjoint non-empty open sets X_1 and X_2 of X such that $X_1 \cup X_2 = X$. Then $O(X_1) \cap E(Y, X, \omega)$ and $O(X_2) \cap E(Y, X, \omega)$ is a separation of $E(Y, X, \omega)$.

Corollary 2.2—Let Y be an ω -extension of X . Then X is connected if and only if Y is connected.

Consequently, X is connected if and only if βX is connected for each completely regular space X .

Example 2.3—Let X be the subspace $\{r \in R : r \geq 1\} \cup \{r \in R : r \leq -1\}$ of the real line R . Let Y be the one-point compactification of X . Then X is not connected but Y is connected.

So, Y is not ω -extension of X by Corollary 2.2.

Definitions 2.4—Let p be a point of a space X . The component of p in X is the maximal connected set of X containing p . The quasicomponent of p in X is the intersection of all clopen sets of X containing p . X is said to be locally connected at p if every neighbourhood of p contains an open connected neighbourhood of p . X is said to be weakly locally connected (connected in kleinen) at p if every neighbourhood of p contains a connected neighbourhood of p ; equivalently for every neighbourhood U of p , the component of p in U is a neighbourhood of p . Finally, X is said to be quasilocally connected at p if for every neighbourhood U of p , the quasicomponent of p in U is a neighbourhood of p (cf. de Groot and McDowell⁶).

Remarks 2.5 : If X is locally connected at p , then X is weakly locally connected at p . Also, if X is weakly locally connected at p , then X is quasilocally connected at p . Furthermore the next properties are equivalent;

- (1) X is locally connected at each point of X ,
- (2) X is weakly locally connected at each point of X ,
- (3) X is quasilocally connected at each point of X .

Theorem 2.6—Let Y be an extension of X . Then $E(Y, X, \omega)$ is not quasilocally connected at any point of $E(Y, X, \omega) - E(Y, X)$.

PROOF : Let p be a point of $E(Y, X, \omega) - E(Y, X)$. Then there is a cellular family $\{U_\alpha\}_{\alpha \in A}$ of open sets of X such that p is in $O(\bigcup_{\alpha \in A} U_\alpha)$ but not in $O(U_\alpha)$ for each $\alpha \in A$. Since p is in $E(Y, X, \omega)$, $\{\alpha \in A : U_\alpha \text{ is non-empty}\}$ is infinite. Let Q be the quasicomponent of p in $O(\bigcup_{\alpha \in A} U_\alpha) \cap E(Y, X, \omega)$. Then for each β of A , $(O(U_\beta) \cup O(\bigcup_{\alpha \in A - \{\beta\}} U_\alpha)) \cap E(Y, X, \omega) = O(\bigcup_{\alpha \in A} U_\alpha) \cap E(Y, X, \omega)$ holds by Lemma 1.2. Hence Q is disjoint from $O(U_\beta)$ for each $\beta \in A$. So, Q is contained in $E(Y, X, \omega) - X$. But since X is dense in $E(Y, X, \omega)$, Q is not a neighbourhood of p in $E(Y, X, \omega)$. Hence $E(Y, X, \omega)$ is not quasilocally connected at p .

Theorem 2.7 (de Groot and McDowell⁶, 2.8)—Let Y be an extension of X . If X is locally connected at a point p of X , then Y is locally connected at p .

In a similar way, one can prove the next theorems, but the proofs are straightforward.

Theorem 2.8—Let Y be an extension of X and p a point of X . Then the following hold;

- (1) if X is weakly locally connected at p , then Y is weakly locally connected at p ,
- (2) if X is quasilocally connected at p , then Y is quasilocally connected at p .

Theorem 2.9—Let Y be an extension of X and p a point of X . If $E(Y, X, \omega)$ is quasilocally connected at p , then X is quasilocally connected at p (consequently, Y is quasilocally connected at p by Theorem 2.8.)

PROOF : Let U be an open neighbourhood of p in $E(Y, X, \omega)$ and Q be the quasicomponent of p in U . We will show that $Q \cap X$ is the quasicomponent of p in $U \cap X$. Suppose not, then there is a clopen set W in $U \cap X$ containing p such that $Q \cap X - W$ is non-empty. Let V be the set $(U \cap X) - W$. Then the following holds:

$$\begin{aligned} U &\subset O(W \cup V) \cap E(Y, X, \omega) \\ &= \{O(W) \cup O(V)\} \cap E(Y, X, \omega) \\ &= \{O(W) \cap E(Y, X, \omega)\} \cup \{O(V) \cap E(Y, X, \omega)\}. \end{aligned}$$

This shows that Q is not quasicomponent of p in U . This is a contradiction

In a similar way one can prove the next theorem.

Theorem 2.10—Let Y be an extension of X and p a point of X . If $E(Y, X, \omega)$ is locally connected at p , then X is locally connected at p (consequently, Y is locally connected at p by Theorem 2.7).

Remark 2.11 As compared with Theorem 2.9 or Theorem 2.10, naturally we have a question as follows;

if $E(Y, X, \omega)$ is weakly locally connected at a point p of X ,
then is X weakly locally connected at p ?

But the answer is negative. Because there exists a completely regular space X and a point p of X at which X is quasilocally connected but not weakly locally connected⁶. According to Theorem 2.9, βX is quasilocally connected at p . But since the quasicomponent and the component coincide in every compact space, we can prove the pointwise quasilocal connectedness and the pointwise weakly local connectedness are the same concept in locally compact spaces. Hence βX is weakly locally connected at p .

Corollary 2.12—Let Y be an extension of X and p a point of X . If $E(Y, X, \omega)$ is (quasi) locally connected at p , then every extension of X , which is contained in Y , is (quasi) locally connected at p .

PROOF : Use Theorems 2.7, 2.8, 2.9 and 2.10.

Corollary 2.13—Let Y be an extension of X and p a point of X . If Z is an extension of X which contained in $E(Y, X, \omega)$, then $E(Y, X, \omega)$ is (quasi) locally connected at p if and only if Z is (quasi) locally connected at p .

PROOF : Because $E(Y, Z, \omega) = E(Y, X, \omega)$ by Lemma 1.23 (1).

Remark 2.14 : According to 2.9 or 2.10, if Y is an extension of X and $E(Y, X, \omega)$ is (quasi) locally connected at a point p of X , then Y is (quasi) locally connected at p . But the converse is not true. Because, the real line R is (quasi) locally connected at every point of R , but the set Q of all of the rational numbers is not (quasi) locally connected at any point of Q . Furthermore $E(R, Q, \omega) = Q$. Because, for each r of $R - Q$, consider $U_1 = \{q \in Q : q < r\}$ and $U_2 = \{q \in Q : r > q\}$.

Theorem 2.15—Let Y be an extension of a locally connected space X . Then $E(Y, X)$ is locally connected. Where a space is said to be locally connected if it is locally connected at every point.

PROOF : Let p be a point of $E(Y, X)$ and W an open set in Y containing p . Since Y is regular, there exists an open neighbourhood V of p in Y whose closure in Y is contained in W . Since X is locally connected at every point, each component of $V \cap X$ is open in X . Let $\{C_\alpha\}_{\alpha \in A}$ be the family of all components of $V \cap X$. Then there exists an α of A such that p is in $O(C_\alpha)$, since p is in $E(Y, X)$. Since $C_\alpha \subset O(C_\alpha) \cap E(Y, X) \subset \text{cl}_Y C_\alpha$ holds, $O(C_\alpha) \cap E(Y, X)$ is a connected open neighbourhood of p in $E(Y, X)$ which is contained in $W \cap E(Y, X)$. Hence $E(Y, X)$ is locally connected.

One can prove the next theorem following Theorems 2.6, 2.7, 2.10 and 2.15.

Theorem 2.16—Let Y be an extension of X . Then X is locally connected if and only if $E(Y, X, \omega)$ is locally connected at each point of $E(Y, X)$ but not at each point of $E(Y, X, \omega) - E(Y, X)$.

The next corollary is also proved using Theorems 2.16, 1.5 and 1.11.

The second half of the next corollary is known as Henriksen and Isbell's theorem³.

Corollary 2.17—Let Y be an ω -extension of X . Then X is locally connected if and only if Y is locally connected at each point of $E(Y, X)$ but not at each point of $Y - E(Y, X)$.

Consequently, for each completely regular space X , X is locally connected if and only if βX is locally connected at each point of μX but not at each point of $\beta X - \mu X$.

Remark 2.18: Unfortunately even if a space X is locally connected, one can not decide whether an extension Y of X is locally connected at a point outside $E(Y, X, \omega)$ only using the local connectedness of X . For example, let $R^* = R \cup \{p\}$ be the one-point compactification of the real line R . Then R^* is locally connected at p and $E(R^*, R, \omega) = R$. Next let $N^* = N \cup \{p\}$ be the one-point compactification of the set N of all natural numbers. Then N^* is not locally connected at p but $E(N^*, N, \omega) = N$. Of course N is locally connected as also R .

Remark 2.19: It is well known that if X is locally connected and pseudocompact space, then each extension of X is locally connected. Hence as compared with Theorems 1.25 and 2.16, it seems that each extension of a pseudocompact space is an ω -extension. But it is not true! Because, let X be a pseudocompact space with $|\beta X - X| \geq 2$. Let x_1 and x_2 be distinct points of $\beta X - X$. Define a function f from βX to $\alpha X = X - \{x_2\}$ such that $f(x) = (x)$ if $x \neq x_2$ but $f(x_2) = x_1$. Let αX be topologized as a quotient space of βX with respect to f . Then $f^{-1}(x_1) = \{x_1, x_2\}$, hence $f^{-1}\{x_1\}$ is not connected. Therefore αX is not ω -extension, according to corollary 1.15 (2). But X is pseudocompact.

In the next two results, we will give a condition for an extension Y of X to be locally connected at some point of $Y - E(Y, X, \omega)$.

Lemma 2.20—Let Y be an extension of a locally connected space X , and U be an open set in Y containing a point p of Y . Furthermore let $\{C_\alpha\}_{\alpha \in A}$ be the family of all components of $U \cap X$, and A_0 be the subset $\{\alpha \in A : p \notin \text{cl}_Y C_\alpha\}$ of A .

If $p \notin \text{cl}_Y (\bigcup_{\alpha \in A_0} C_\alpha)$, then p has a connected open neighbourhood which is contained in U .

PROOF : For $\alpha \in A - A_0$, $U \cap (\text{cl}_Y C_\alpha - \text{cl}_Y (\bigcup_{\beta \in A_0} C_\beta))$ is a connected set containing p , since C_α is connected. Therefore $U \cap (\bigcup_{\alpha \in A - A_0} \text{cl}_Y C_\alpha - \text{cl}_Y (\bigcup_{\beta \in A_0} C_\beta))$ is a connected set. Furthermore the following holds;

$$\begin{aligned} & \bigcup_{\alpha \in A - A_0} C_\alpha \\ & \subset U \cap (\bigcup_{\alpha \in A - A_0} \text{cl}_Y C_\alpha - \text{cl}_Y (\bigcup_{\beta \in A_0} C_\beta)) \\ & \subset U - \text{cl}_Y (\bigcup_{\beta \in A_0} C_\beta) \\ & \subset \text{cl}_Y (\bigcup_{\alpha \in A - A_0} U \cap C_\alpha). \end{aligned}$$

Hence $U - \text{cl}_Y (\bigcup_{\beta \in A_0} C_\beta)$ is a connected open neighbourhood in Y containing p .

Remark 2.21 : If Y is an extension of a locally connected space X and p is a point of $E(Y, X)$, then for each open neighbourhood U of p in Y , p is not in cl_Y

$(\bigcup_{\alpha \in A_0} C_\alpha)$; where all notations are as in Lemma 2.20. Because, there is an $\alpha \in A - A_0$ such that p is in $O(C_\alpha)$. Therefore p has a connected open neighbourhood base in Y .

But for pseudocompact case, Lemma 2.20 gives full play for points of $Y - E(Y, X, \omega)$.

Theorem 2.22(Doss³, 2.4; Walker^{9.7})—If Y is an extension of a locally connected pseudocompact space X , then Y is locally connected.

PROOF : Let U be an open neighbourhood of a point p in Y , and $\{C_\alpha\}_{\alpha \in A}$ be the family of all components of $U \cap X$. Let V be an open neighbourhood of p whose closure in Y is contained in U . Since X is pseudocompact, for finite α of A , $(V \cap X) \cap C_\alpha$ is non-empty. Hence the condition of Lemma 2.20 is satisfied.

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A CHARACTERIZATION OF SEMIGROUPS ADMITTING A UNIQUE MULTIPLICATIVE INVARIANT MEAN

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Abelian semigroups admitting a unique left invariant mean were characterized by Luthar [*Trans. Am. Math. Soc.* 104 (1962), 403-11]. They are precisely those semigroups admitting a finite two sided ideal which is a group. Klawe [*Trans. Am. Math. Soc.* 231 (1977), 507-18] has extended the result for non abelian semigroups also. In this paper semigroups admitting a unique multiplicative left invariant mean are characterized.

PRELIMINARIES AND NOTATIONS

For any semigroup S , let $m(S)$ be the Banach space of bounded real valued functions on S with the sup norm. For each $s \in S$, we define a linear operator $l_s (r_s)$ on $m(S)$ by $l_s f(t) = f(st)$ $r_s f(t) = f(ts)$ for every $t \in S$. A mean on S is a positive element of norm one in the dual $m(S)^*$ of $m(S)$. We say that $\mu \in m(S)^*$ is left (right) invariant if $\mu(l_s f) = \mu(f)$ ($\mu(r_s f) = \mu(f)$) for each $f \in m(S)$ and $s \in S$. A semigroup S is said to be left (right) amenable if it has a left (right) invariant mean, and we denote the set of left (right) invariant means on $m(S)$ by $ML(S)$ ($Mr(S)$). When S is both left and right amenable we say that S is amenable. By $\dim \langle ML(S) \rangle$ ($\dim \langle Mr(S) \rangle$) we mean the dimension of the linear space spanned by the set $ML(S)$ ($Mr(S)$) in $m(S)^*$.

A mean μ is said to be multiplicative if $\mu(fg) = \mu(f) \mu(g)$ for all $f, g \in m(S)$. A semigroup is called extremely left amenable, denoted as ELA, if it has a multiplicative left invariant mean. For a detailed account of properties of ELA semigroups we refer to Granirer^{1,2}.

For any subset $A \subset S$ let, I_A denote its characteristic function, that is, $I_A(s) = 1$ if $s \in A$ and $I_A(s) = 0$ if $s \notin A$. If A is a subset of S , we use $|A|$ to denote the cardinality of A .

A subset T of S is called left thick³ if for every finite subset F of S , there is an element s in S such that $Fs \subset T$. A subset A of S , is called strongly left thick³ if for every subset B of S with $|B| < |A|$, $A - B$ is left thick. Klawe³ (Theorem 2.2) has established certain interesting results regarding these thick subsets and these results were used to compute the cardinality of the multiplicative left invariant means on $m(S)$.

CHARACTERIZATION

Proposition 1—Let S be a semigroup and μ be a multiplicative left invariant mean on S . Let $\mathcal{F} = \{E \subset S : \mu(1_E) = 1\}$ and let $A \in \mathcal{F}$ be such that $|A| \leq |E|$ for all $E \in \mathcal{F}$. Then A is strongly left thick, in particular, there exist strongly left thick sets A such that $\mu(1_A) = 1$.

PROOF : Let B be a subset of A such that $|B| < |A|$. Then $B \notin \mathcal{F}$ and consequently $\mu(1_B) < 1$. Since μ is multiplicative and $1_B^2 = 1_B$, this implies that $\mu(1_B) = 0$. Hence $\mu(1_{A-B}) = \mu(1_A) - \mu(1_B) = 1$. Therefore $A - B$ is left thick (Theorem 7 of Mitchell⁶). Hence A is strongly left thick.

Theorem 2—Let S be an ELA semigroup. Then the multiplicative left invariant mean on S is unique if and only if every nonempty left thick set is a singleton.

PROOF : Suppose that S admits a unique multiplicative left invariant mean, say μ . By Theorem 6 of Granier², μ is the only extreme point of the (convex) set of all left invariant means; by Krein-Milman theorem this implies that μ is the only left invariant mean on S .

Now let A be any strongly left thick subset of S . Then A can be expressed as a union of a family $\{D_\nu : \nu \in \Gamma\}$ of pairwise disjoint left thick subsets, where Γ is an indexing set such that $|\Gamma| = |A|$ (Theorem 2.2 of Klawe³). For each left thick subset D there exists a left invariant mean λ such that $\lambda(1_D) = 1$. Theorem 7 of Mitchell⁶. Since μ is the only left invariant mean on S this implies $\mu(1_{D_\nu}) = 1$ for all $\nu \in \Gamma$. As $D_\nu, \nu \in \Gamma$ are pairwise disjoint, this is impossible unless $|\Gamma| = 1$. Hence $|A| = 1$.

Conversely, suppose that every strongly left thick subset is a singleton. Let μ and ν be two multiplicative left invariant means on S . By Proposition 1 there exist strongly left thick subsets A and B such that $\mu(1_A) = \nu(1_B) = 1$. By hypothesis $A = \{a\}$ and $B = \{b\}$ for some $a, b \in S$. Let $D = \{a, b\} = A \cup B$. If E is any subset of D such that $|E| < |D|$ then clearly $D - E$ contains either A or B and therefore $D - E$ is left thick. This means that D is strongly left thick. By assumption this implies that D is a singleton; in other words, $a = b$ and consequently $\mu = \nu$.

Theorem 3—A semigroup S admits a unique multiplicative left invariant mean if and only if it admits a zero.

PROOF : Suppose S has a zero, say 0 . Then $f \rightarrow f(0)$ is obviously a multiplicative left invariant mean. Further, if μ is any multiplicative left invariant mean then for any $f \in m(S)$, $\mu(f) = \mu(l_0 f) = f(0)$, since $l_0 f$ is the constant function $f(0)$. This proves uniqueness.

Conversely suppose that S admits a unique multiplicative left invariant mean, say μ . By Proposition 1 there exists a strongly left thick subset A such that $\mu(1_A) = 1$ and by Theorem 2, A is a singleton, say $\{a\}$. Thus $\mu(1_{\{a\}}) = 1$, which means that for any

$f \in m(S)$, $\mu(f) = f(a)$. Then for any $x \in S$ and $f \in m(S)$ we get $f(xa) = (l_x f)(a) = \mu(l_x f) = \mu(f) = f(a)$. This implies that $xa = a$ for all $x \in S$. For all $x \in S$ and $f \in m(S)$ define $\lambda_x(f) = \mu(r_x f)$. It is straightforward to verify that λ_x is a multiplicative left invariant mean on S , so that by uniqueness of μ we have $\lambda_x = \mu$ for all $x \in S$. Hence for all $f \in m(S)$, $f(ax) = r_x f(a) = \mu(r_x f) = \lambda_x(f) = \mu(f) = f(a)$. This implies that $ax = a$ for all $x \in S$. Thus a is a zero on S .

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FRACTIONAL INTEGRAL OPERATORS AND THE GENERALIZED HYPERGEOMETRIC FUNCTIONS

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In the present paper we derive a number of new and useful results for the fractional integral operators involving the generalized hypergeometric functions. First we obtain elegant expressions for the composition of these operators. It is shown how these operators can be closely related with the algebra of functions having Mellin convolution as the product. Inversion formulas and the relations of our operators with the generalized Hankel transforms are also established. The main results of this paper provide us the extensions to the earlier results due to Erdélyi³, Saxena and Kumbhat⁸ and others.

1. INTRODUCTION

The object of this paper is to establish a number of key (new and interesting) formulas for the fractional integral operators defined below :

$$\left[I_{\alpha}^h : f(x) \right] \equiv I_{\alpha}^h f(x) = x^{-h-\alpha} R_{\alpha} x^h f(x) \quad \dots(1.1)$$

$$\left[K_{\beta}^{\lambda} : f(x) \right] \equiv K_{\beta}^{\lambda} f(x) = x^{\lambda} W_{\beta} x^{-\lambda-\beta} f(x) \quad \dots(1.2)$$

where

$$R_{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} {}_1F_0 \left[(a); (b); z \left(1 - \frac{s}{x} \right) \right] f(s) ds, \\ \{ \text{Re}(\alpha) > 0, \text{Re}(\epsilon) > -1 \} \quad \dots(1.3)$$

and

$$W_{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_x^{\infty} (s-x)^{\beta-1} {}_1F_0 \left[(c); (d); z' \left(1 - \frac{x}{s} \right) \right] f(s) ds, \\ \{ \text{Re}(g) > \text{Re}(\beta) > 0 \}. \quad \dots(1.4)$$

Here (a) abbreviates the array of A -parameters a_1, \dots, a_A and so on, ${}_A F_B [z]$ stands for the generalized hypergeometric function. Also we have made the assumption that $f(x) \in C'$, where C' denotes the class of functions $f(x)$ which are such that

$$f(x) = \begin{cases} 0(x^e), & |x| \rightarrow 0 \\ 0(x^{-g}), & |x| \rightarrow \infty. \end{cases}$$

The operators (1.1) and (1.2) also exist under the following alternative conditions

- (i) $1 \leq p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$
- (ii) $\operatorname{Re}(h) > -\frac{1}{q}, \operatorname{Re}(\alpha) > -\frac{1}{q}, \operatorname{Re}(\beta) > -\frac{1}{q}, \operatorname{Re}(\lambda) > -\frac{1}{p},$
- (iii) $A \leq B$ or $A = B + 1$ and $|z| < 1$
- (iv) $C \leq D$ or $C = D + 1$ and $|z'| < 1$
- (v) $f(x) \in L_p(0, \infty).$

Under these conditions both $I_\alpha^h f(x)$ and $K_\beta^\lambda f(x)$ exist and also both belong to $L_p(0, \infty)$.

Remark : The operators (1.1) and (1.2) can be considered as the generalizations of the well-known Riemann-Liouville and Weyl fractional integrals and Erdélyi-Kober operators. Though these operators are special cases of the operators studied by Saxena and Kumbhat⁷, yet they are also sufficiently general in nature and unify a number of other fractional integral operators introduced in the literature from time to time. In this paper we shall establish certain new results, in a very neat and compact form, for our operators. The importance of our results lies in the fact that these can not be established for Saxena and Kumbhat's operators in terms of the named special function. Moreover, many important properties such as commutativity and associativity which are true for our operators may not hold for Saxena and Kumbhat's operators.

2. COMPOSITIONS OF FRACTIONAL OPERATORS DEFINED BY (1.1) AND (1.2)

In view of definition (1.1), we have

$$\begin{aligned} I_\alpha^h \left(I_\beta^\lambda f(x) \right) &= \frac{x^{-h-\alpha}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x (x-s)^{\alpha-1} {}_A F_B \left[(a); (b); z \left(1 - \frac{s}{x} \right) \right] s^{h-\lambda-\beta} \\ &\quad \times \left(\int_0^s (s-u)^{\beta-1} {}_A F_B \left[(a); (b); z \left(1 - \frac{u}{s} \right) \right] u^\lambda f(u) du \right) ds. \end{aligned} \quad \dots(2.1)$$

Since $f(x) \in C'$, the change of integrations is guaranteed when

- (i) $A \leq B; A = B + 1$ and $|z| < 1,$
- (ii) $\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0, \operatorname{Re}(\lambda + e) > -1,$

and by using the well-known Fubini's theorem we have

$$I_{\alpha}^h \left(I_{\beta}^{\lambda} f(x) \right) = \frac{x^{-h-\alpha}}{\Gamma(\alpha) \Gamma(\beta)} \int_0^x u^{\lambda} f(u) \Delta du \quad \dots(2.2)$$

where

$$\Delta = \int_u^x (x-s)^{\alpha-1} (s-u)^{\beta-1} s^{h-\lambda-\beta} {}_A F_B \left[(a); (b); z \left(1 - \frac{s}{x} \right) \right] {}_A F_B \left[(a); (b); z \left(1 - \frac{u}{s} \right) \right] ds. \quad \dots(2.3)$$

For the determination of the value of Δ , use series representation for ${}_A F_B$ -functions, change the order of summation and integration, make the substitution $w = (x-s)/(x-u)$ and use the known results (Oberhettinger⁶ (p. 19, eqn. (2.33)) therein. Substituting the value of Δ , so obtained in (2.2), we arrive at the result :

$$\begin{aligned} I_{\alpha}^h \left(I_{\beta}^{\lambda} f(x) \right) &= \frac{x^{-\lambda-\alpha-\beta}}{\Gamma(\alpha+\beta)} \int_0^x (x-u)^{\alpha+\beta-1} u^{\lambda} f(u) \\ &\times F^{(3)} \left[\begin{matrix} - : -; \lambda + \beta - h; \alpha : (a); (a), \beta : -; \\ \alpha + \beta : -; - : -; (b); (b), \lambda + \beta - h : -; \end{matrix} \right. \\ &\left. z \left(1 - \frac{u}{x} \right), z \left(1 - \frac{u}{x} \right), \left(1 - \frac{u}{x} \right) \right] du \quad \dots(2.4) \end{aligned}$$

where

$$\begin{aligned} F^{(3)} [x, y, z] &\equiv F^{(3)} \left[\begin{matrix} (a) : : (b); (b'); (b'') : (c); (c'); (c'') \\ (e) : : (g); (g'); (g'') : (h); (h'); (h''); \end{matrix} \quad x, y, z \right] \\ &= \sum_{m,n,p=0}^{\infty} A(m, n, p) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \quad \dots(2.5) \end{aligned}$$

is the general triple hypergeometric series defined by Srivastava⁹ (see also Srivastava and Manocha¹⁰, (p. 89)). The value of $A(m, n, p)$ and other details of this function can be found in the paper and book referred to above.

If we use the series representation (2.5) for $F^{(3)}$ -function and one of the Euler's transformation of the hypergeometric series (Erdélyi *et al.*⁴, p. 64, eqn. (23)), we find that

$$(u/x)^{\lambda} F^{(3)} \left[\begin{matrix} - : -; \lambda + \beta - h; \alpha : (a); (a), \beta : -; \\ \alpha + \beta : -; - : -; (b); (b), \lambda + \beta - h : -; \end{matrix} \right]$$

(equation continued on p. 254)

$$\begin{aligned}
& z \left(1 - \frac{u}{x} \right), z \left(1 - \frac{u}{x} \right), \left(1 - \frac{u}{x} \right) \Bigg] \\
& = (u/x)^h F^{(3)} \left[\begin{matrix} - : -; h + \alpha - \lambda; \beta; (a); (a), \alpha; -; \\ \alpha + \beta : -; -; -; (b); (b), h + \alpha - \lambda; -; \end{matrix} \right. \\
& \quad \left. z \left(1 - \frac{u}{x} \right), z \left(1 - \frac{u}{x} \right), \left(1 - \frac{u}{x} \right) \right] \quad \dots(2.6)
\end{aligned}$$

(2.6) shows that the operators I_α^h and I_β^λ commute.

On proceeding with similar computations, the expression for the product of the form (2.4) involving the operators (1.2) can be easily derived. This is given by

$$\begin{aligned}
K_\alpha^h \left(K_\beta^\lambda f(x) \right) &= K_\beta^\lambda \left(K_\alpha^h f(x) \right) = \frac{x^h}{\Gamma(\alpha + \beta)} \int_x^\infty (u-x)^{\alpha+\beta-1} u^{-\lambda-\beta-h} f(u) \\
&\times F^{(3)} \left[\begin{matrix} - : -; h + \alpha - \lambda; \beta; (c); (c), \alpha; -; \\ \alpha + \beta : -; -; -; (d); (d), h + \alpha - \lambda; -; \end{matrix} \right. \\
&\quad \left. z' \left(1 - \frac{x}{u} \right), z' \left(1 - \frac{x}{u} \right), \left(1 - \frac{x}{u} \right) \right] \quad \dots(2.7)
\end{aligned}$$

Next, we obtain the representation of the composition of operators (1.1) and (1.2). From (1.1) and (1.2), we have

$$\begin{aligned}
I_\alpha^h \left(K_\beta^\lambda f(x) \right) &= \frac{x^{-h-\alpha}}{\Gamma(\alpha) \Gamma(\beta)} \int_0^x (x-s)^{\alpha-1} {}_A F_B \left[(a); (b); z \left(1 - \frac{s}{x} \right) \right] s^{\lambda+h} \\
&\times \left(\int_s^\infty (u-s)^{\beta-1} {}_C F_D \left[(c); (d); z' \left(1 - \frac{s}{u} \right) \right] u^{-\lambda-\beta} f(u) du \right) ds. \quad \dots(2.8)
\end{aligned}$$

On changing the order of integration which is justified under the conditions that $f(x) \in C'$ and $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 0$, $\text{Re}(\lambda + h) > 0$, $\text{Re}(\lambda + h + 1) > 0$ and Fubini's theorem, we find that (2.8) takes the form

$$I_\alpha^h \left(K_\beta^\lambda f(x) \right) = \int_0^\infty \theta(u, x) f(u) du \quad \dots(2.9)$$

where

$$\begin{aligned}
\theta(u, x) &= \frac{x^{-h-\alpha} u^{-\lambda-\beta}}{\Gamma(\alpha) \Gamma(\beta)} \int_0^{\min(x, u)} s^{\lambda+h} (x-s)^{\alpha-1} (u-s)^{\beta-1} \\
&\times {}_A F_B \left[(a); (b); z \left(1 - \frac{s}{x} \right) \right] {}_C F_D \left[(c); (d); z' \left(1 - \frac{s}{u} \right) \right] ds. \quad \dots(2.10)
\end{aligned}$$

The above integral can be evaluated under the cases (i) $x < u$ and (ii) $x > u$ and with the help of simple substitutions.

In a similar manner, the product (2.9) taken in the other order i.e. $K_{\beta}^{\lambda} \left(I_{\alpha}^h f(x) \right)$ can be computed. It is easy to observe that this product has a similar representation as mentioned in (2.9), where, indeed, the inner integral corresponding to (2.10) is markedly different. If $\phi(u, x)$ denotes this integral, then the following functional relation exists between θ and ϕ

$$(u x) \phi(u, x) = \theta(x^{-1}, u^{-1}). \quad \dots(2.11)$$

The final result that emerges is given by

$$\begin{aligned} I_{\alpha}^h \left(K_{\beta}^{\lambda} f(x) \right) &= K_{\beta}^{\lambda} \left(I_{\alpha}^h f(x) \right) = \frac{\Gamma(1+h+\lambda) x^{-h-1}}{\Gamma(\alpha) \Gamma(1+h+\lambda+\beta)} \\ &\quad \int_0^x u^h \left(1 - \frac{u}{x} \right)^{\alpha+\beta-1} f(u) \\ &\quad F^{(3)} \left[\begin{matrix} \alpha + \beta + \lambda + h : - ; \beta ; - ; (a); (c) ; - ; \\ - : : \alpha + \beta + \lambda + h ; 1 + \beta + \lambda + h ; - ; (b); (d) ; - ; \end{matrix} \right. \\ &\quad \left. z \left(1 - \frac{u}{x} \right), z' \left(1 - \frac{u}{x} \right), \frac{u}{x} \right] du \\ &\quad + \frac{\Gamma(1+h+\lambda) x^{\lambda}}{\Gamma(\beta) \Gamma(1+h+\lambda+\alpha)} \int_x^{\infty} u^{-\lambda-1} \left(1 - \frac{u}{x} \right)^{\alpha+\beta-1} \\ &\quad F^{(3)} \left[\begin{matrix} \alpha + \beta + \lambda + h : - ; - ; \alpha : (a); (c) ; - ; \\ - : : \alpha + \beta + \lambda + h ; - ; 1 + \alpha + \lambda : (b); (d) ; - ; \end{matrix} \right. \\ &\quad \left. z \left(1 - \frac{x}{u} \right), z' \left(1 - \frac{x}{u} \right), \frac{x}{u} \right] f(u) du. \quad \dots(2.12) \end{aligned}$$

Evidently the results (2.4), and (2.7) and (2.12) provide us the results of (Erdélyi³ pp. 166-167, eqns (6.1) - (6.3); see also, Buschman² (p. 100, eqn. (2.4)) as special cases when we take $z \rightarrow 0$ and make some simplifications.

3. MELLIN CONVOLUTION, MELLIN TRANSFORM AND INVERSION FORMULAS

We shall use the symbol to denote the Mellin convolution taken in the form

$$(f * g)(x) = \int_0^{\infty} u^{-1} f\left(\frac{x}{u}\right) g(u) du. \quad \dots(3.1)$$

If $f, g \in L_p(0, \infty)$ then $(f * g) \in L_p(0, \infty)$.

Let us define a function by

$$I_{\alpha, h}(x) = \frac{1}{\Gamma(\alpha)} x^{-h-\alpha} (x-1)^{\alpha-1} H(x-1) {}_1F_1 \left[(a); (b); z \left(\frac{x-1}{x} \right) \right] \quad \dots(3.2)$$

where $H(z)$ denotes the Heaviside unit function. Then we can represent the operator (1.1) in the form of convolution (3.1). Indeed we have

$$\begin{aligned} I_{\alpha}^h f(x) &= \int_0^{\infty} s^{-1} \frac{1}{\Gamma(\alpha)} \left\{ \left(\frac{x}{s} \right)^{-h-\alpha} \left(\frac{x}{s} - 1 \right)^{\alpha-1} H \left(\frac{x}{s} - 1 \right) \right. \\ &\quad \left. {}_1F_1 \left[(a); (b); z \left(\frac{\frac{x}{s} - 1}{\frac{x}{s}} \right) \right] \right\} f(s) ds \\ &= (I_{\alpha,h} * f)(x) \end{aligned} \quad \dots (3.3)$$

where $I_{\alpha,h}(x) \in L_p(0, \infty)$ for $\alpha > 0, h > 0$.

This reveals that the properties of commutativity and associativity hold true for the operator (1.1), since these can be identified with the elements of algebra of functions having the Mellin convolution (3.1) as the product.

In a similar manner if we define

$$K_{\beta,\lambda}(x) = \frac{1}{\Gamma(\beta)} (1-x)^{\beta-1} H(1-x) {}_1F_1 \left[(c); (d); z'(1-x) \right] \quad \dots (3.4)$$

then

$$K_{\beta}^{\lambda} f(x) = (K_{\beta,\lambda} * f)(x) \quad \dots (3.5)$$

where

$$K_{\beta,\lambda}(x) \in L_p(0, \infty) \text{ for } \beta > 0, \lambda > -1.$$

From the definition of Mellin transform and the results [Erdélyi *et al*⁵, p. 311, eqn. (31), (32)], it follows easily that

$$M_s \{I_{\alpha,h}(x)\} = \frac{\Gamma(1+h-s)}{\Gamma(1+h+\alpha-s)} {}_{A+1}F_{B+1} [(a), \alpha; (b), 1+h+\alpha-s; z] \quad \dots (3.6)$$

$$M_s \{K_{\beta,\lambda}(x)\} = \frac{\Gamma(\lambda+s)}{\Gamma(\beta+\lambda-s)} {}_{C+1}F_{D+1} [(c), \beta; (d), \lambda+\beta+s; z'] \quad \dots (3.7)$$

where $M_s \{f(x)\}$ stands for the well-known Mellin transform. The above results in conjunction with (3.3) and (3.5) yield

$$\begin{aligned} M_s \left\{ I_{\alpha}^h f(x) \right\} &= \frac{\Gamma(1+h-s)}{\Gamma(1+h+\alpha-s)} {}_{A+1}F_{B+1} [(a), \alpha; (b), 1+h+\alpha-s; z] M_s \{f(x)\} \end{aligned} \quad \dots (3.8)$$

$$\begin{aligned} M_s \left\{ K_{\beta}^{\lambda} f(x) \right\} &= \frac{\Gamma(\lambda+s)}{\Gamma(\beta+\lambda+s)} {}_{C+1}F_{D+1} [(c), \beta; (d), \lambda+\beta+s; z'] M_s \{f(x)\}. \end{aligned} \quad \dots (3.9)$$

If we use the well-known Mellin inversion theorem in conjunction with (3.8) and (3.9), we easily arrive at the inversion formulas (given below) for the operators defined by (1.1) and (1.2) respectively.

$$\frac{1}{2} [f(t+0) + f(t-0)] = \frac{1}{2\pi i} \lim_{\gamma \rightarrow \infty} \int_{c'-i\gamma}^{c'+i\gamma} \frac{t^{-s}}{\Phi(s)} M_s \left\{ I_{\alpha}^h f(x) \right\} ds \quad \dots(3.10)$$

where

$$\Phi(s) = \frac{\Gamma(1+h-s)}{\Gamma(1+h+\alpha-s)} {}_{A+1}F_{B+1} [(a), \alpha; (b), 1+\alpha+h-s; z].$$

The formula (3.10) is valid under the conditions

- (i) $f(t) \in L_p(0, \infty)$, $1 \leq p \leq 2$
- (ii) $f(t)$ is of bounded variation at the point $t = x$ ($x > 0$).

$$\frac{1}{2} [f(t+0) + f(t-0)] = \frac{1}{2\pi i} \lim_{\gamma \rightarrow \infty} \int_{c'-i\gamma}^{c'+i\gamma} \frac{t^{-s}}{\Omega(s)} M_s \left\{ K_{\beta}^{\lambda} f(x) \right\} ds \quad \dots(3.11)$$

where

$$\Omega(s) = \frac{\Gamma(\lambda+s)}{\Gamma(\beta+\lambda+s)} {}_{C+1}F_{D+1} [(c), \beta; (d), \lambda+\beta+s; z']. \quad \dots(3.12)$$

The formula (3.11) is valid under the same conditions as mentioned with (3.10). Also when $f(t)$ is continuous at $t = x$ ($x > 0$) then the left hand sides of (3.10) and (3.11) are equal to $f(t)$.

If we take $A = C = 2$, $B = D = 1$, $b_1 = \alpha$, $d_1 = \beta$, $z = z' = 1$ in (3.10) and (3.11), we arrive at the inversion formulas established earlier by Saxena and Kumbhat⁸ [p. 142, eqn. (3.1) and (3.3)].

4. RELATIONS BETWEEN OPERATORS (1.1) AND (1.2) AND THE GENERALIZED HANKEL TRANSFORM

The generalized Hankel transform is defined by the equation¹

$$H_{\lambda}^{\mu} \{f: z\} = \int_0^{\infty} (zt)^{\nu} J_{\lambda}^{\mu}(zt) f(t) dt, \quad (z > 0) \quad \dots(4.1)$$

where $J_{\lambda}^{\mu}(x)$ is the Bessel-Wright function [see Srivastava and Manocha¹⁰, (p. 51, eqn. (22))].

In this section we shall establish the following two new theorems giving us the interconnections between our operators defined by (1.1), (1.2) and the generalized Hankel transforms :

Theorem 1—If f and $H_{\sigma}^{\mu} \{f: u\}$ are related to $L_p(0, \infty)$ and if $u > 0, \mu > 0, \operatorname{Re}(\alpha) > 0, \operatorname{Re}[(\sigma + 1)/\mu] - \nu + h > 0$, then

$$I_{\alpha}^h f(x) = \int_0^{\infty} \phi(x, u) H_{\sigma}^{\mu} \{f: u\} du \quad \dots(4.2)$$

where

$$\begin{aligned} \phi(x, u) &= (ux)^{[(\sigma+1)/\mu] - \nu + \alpha + h - 1} \\ &\sum_{r=0}^{\infty} \frac{\Gamma[(\sigma + r + 1)/\mu] - \nu + h}{\Gamma[(\sigma + r + 1)/\mu] - \nu + \alpha + h} \frac{(-ux)^r / \mu}{\Gamma[(\sigma + r + 1)/\mu] r!} \\ &\times {}_{A+1}F_{B+1}[(a), \alpha; (b), \{(\sigma + r + 1)/\mu\} - \nu + \alpha + h; z], \end{aligned} \quad \dots(4.3)$$

provided that $A \leq B$ or $A = B + 1, |z| < 1$.

PROOF : We have¹

$$f(t) = \frac{1}{\mu} \int_0^{\infty} (tu)^{[(\sigma+1)/\mu] - \nu - 1} J_{[(\sigma+1)/\mu] - 1}^{1/\mu} (tu)^{1/\mu} H_{\sigma}^{\mu} \{f: u\} du \quad \dots(4.4)$$

therefore

$$\begin{aligned} I_{\alpha}^h f(x) &= \frac{1}{\mu} \int_0^{\infty} u^{[(\sigma+1)/\mu] - \nu - 1} I_{\alpha}^h \{x^{[(\sigma+1)/\mu] - \nu - 1} J_{[(\sigma+1)/\mu] - 1}^{1/\mu} (ux)^{1/\mu}\} \\ &\times H_{\sigma}^{\mu} \{f: u\} du. \end{aligned} \quad \dots(4.5)$$

The change of order of integration is justified under the conditions given above.

Now evaluating the value of $I_{\alpha}^h \{ \}$ occurring in the integrand of (4.5), we arrive at Theorem 1.

Theorem 2—If f and $H_{\sigma}^{\mu} \{f: u\}$ are related to $L_p(0, \infty)$ and if $u > 0, \mu > 0, \operatorname{Re}(\beta) > 0, \operatorname{Re}[\lambda - \{(\sigma + 1)/\mu\} + \nu + 1] > 0$, then

$$K_{\beta}^{\lambda} f(x) = \int_0^{\infty} \phi'(x, u) H_{\sigma}^{\mu} \{f: u\} du \quad \dots(4.6)$$

where

$$\begin{aligned} \phi'(x, u) &= (ux)^{[(\sigma+1)/\mu] - \nu - 1} \sum_{r=0}^{\infty} \frac{\Gamma[\lambda - \{(\sigma + r + 1)/\mu\} + \nu + 1] (-ux)^r / \mu}{r! \Gamma[(\sigma + r + 1)/\mu] \Gamma[\lambda + \beta - \{(\sigma + r + 1)/\mu\} + \nu + 1]} \\ &\times {}_{C+1}F_{D+1}[(c), \beta; (d), \lambda + \beta - \{(\sigma + r + 1)/\mu\} + \nu + 1; z'] \end{aligned} \quad \dots(4.7)$$

provided that $C \leq D$ or $C = D + 1$, $|z'| < 1$.

The proof of Theorem 2 is similar to that Theorem 1.

Theorems 1 and 2 provide us interesting generalizations of the results established by Saxena and Kumbhat⁸, (Theorems 5 and 6).

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ASSOCIATED KERR-NEWMAN METRIC IN THE COSMOLOGICAL BACKGROUND

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A Kerr-like exact solution of the Einstein-Maxwell field equations is presented. It is interpreted as the associated Kerr-Newman metric in the cosmological background of the closed Robertson-Walker universe. The associated Kerr-Newman metric in the background of the Einstein's static universe is also briefly discussed.

1. INTRODUCTION

We know that the Robertson-Walker metric with positive space-time curvature can be expressed in the form

$$ds^2 = dt^2 - \frac{R^2}{R_0^2} \left[dx^2 + dy^2 + dz^2 + \frac{(xdx + ydy + zdz)^2}{R_0^2 - (x^2 + y^2 + z^2)} \right] \quad \dots(1.1)$$

where R_0 is a constant and R is an arbitrary function of time t . When $R = R_0$, the metric (1.1) becomes the metric of Einstein static universe.

Let us carry out the following transformation from the co-ordinates (x, y, z) to (r, ψ, β)

$$\left. \begin{aligned} x &= R_0 \sin(r/R_0) \sin \psi \cos \beta, \\ y &= R_0 \sin(r/R_0) \sin \psi \sin \beta, \\ z &= R_0 \sin(r/R_0) \cos \psi. \end{aligned} \right\} \quad \dots(1.2)$$

The metric (1.1) reduces to the form

$$ds^2 = dt^2 - \frac{R^2}{R_0^2} \left\{ dr^2 + R_0^2 \sin^2 \left(\frac{r}{R_0} \right) (d\psi^2 + \sin^2 \psi d\beta^2) \right\}. \quad \dots(1.3)$$

Let us define the co-ordinates x and u by the differential relations

$$dx = \frac{R}{R_0} dt, \quad du = \frac{R^2}{R_0^2} dx - dr. \quad \dots(1.4)$$

With the aid of (1.4), one can express the metric (1.3) in the form

$$ds^2 = 2du dx - \frac{R^2}{R_0^2} du^2 - R^2 \sin^2 \left(\frac{r}{R_0} \right) (d\psi^2 + \sin^2 \psi d\beta^2). \quad \dots(1.5)$$

Using the method of complex co-ordinate transformation, a new solution of Einstein's vacuum field equations is obtained by Demianski². The geometry of this solution is described by the line element

$$ds^2 = 2 [du + a \{\sin^2 \psi \log \tan \psi/2 - \cos \psi\} d\beta] dx \\ - (r^2 + Z^2) (d\psi^2 + \sin^2 \psi d\beta^2) \\ - \left(1 + \frac{m}{r^2 + Z^2}\right) [du + a \{\sin^2 \psi \log \tan \psi/2 - \cos \psi\} d\beta]^2 \quad \dots(1.6)$$

where

$$Z = a [1 + \cos \psi \log \tan \psi/2]. \quad \dots(1.7)$$

Here m and a are constants. When $a = 0$ (1.6) becomes the well-known Schwarzschild exterior metric. The space-time described by the metric (1.6) is Petrov type II and possesses a singularity on the rotation axis $\psi = 0$ and $\psi = \pi$. Vaidya⁷ has designated the metric (1.6) as the associated Kerr metric. The electromagnetic generalization of (1.6) has been discussed by Bhatt¹. We shall call this generalization the associated Kerr-Newman metric.

The principal aim of the present article is to derive a new exact solution of Einstein-Maxwell field equations describing the associated Kerr-Newman metric in the background of the Robertson-Walker universe described by (1.5).

It should be noted that the Kerr metric in the cosmological background has been discussed by Vaidya⁶ and the Kerr-Newman metric³ in the cosmological background has been discussed by Patel and Trivedi⁴.

It is easy to see that the metric (1.5) and (1.6) are particular cases of the Kerr-NUT metric discussed by Vaidya *et al.*⁵

We begin with the general Kerr-NUT metric

$$ds^2 = 2 (du + g \sin \alpha d\beta) dx - M^2 (d\alpha^2 + \sin^2 \alpha d\beta^2) \\ - 2L (du + g \sin \alpha d\beta)^2 \quad \dots(1.8)$$

where

$$g = g(\alpha), L = L(u, x, \alpha), M = M(u, x, \alpha).$$

We shall use the method of differential forms for the discussion of our solution. This method is standard now and therefore we shall not enter into the details. Introducing the basic 1-forms

$$\theta^1 = du + g \sin \alpha d\beta, \theta^2 = M d\alpha \\ \theta^3 = M \sin \alpha d\beta, \theta^4 = dx - L \theta^1 \quad \dots(1.9)$$

the metric (1.8) takes the form

$$ds^2 = 2\theta^1\theta^4 - (\theta^2)^2 - (\theta^3)^2 = g_{(ab)} \theta^a \theta^b. \quad \dots(1.10)$$

Now onwards the bracketed indices indicate tetrad components with respect to the tetrad (1.9). Using the tetrad (1.9) Vaidya *et al.*⁵ have obtained the tetrad components $R_{(ab)}$ of the Ricci tensor for the metric (1.8). For the sake of brevity they are not listed here. We shall use the expressions for $R_{(ab)}$ given by Vaidya *et al.*⁵.

2. THE ELECTROMAGNETIC FIELD

Let us consider the electromagnetic field described by the 4-potential A_l given by

$$A_l = (A, 0, A g \sin \alpha, 0) \quad \dots(2.1)$$

where $A = A(u, x, \alpha)$. We have named the co-ordinates as $x^1 = u$, $x^2 = \alpha$, $x^3 = \beta$, $x^4 = x$. The electromagnetic field tensor F_{lk} is defined by $F_{lk} = A_{l,k} - A_{k,l}$. Using (2.1) we can find F_{lk} . We raise and lower the indices with help of the metric (1.8).

The surviving F^{ik} are given by

$$\begin{aligned} -F^{12} &= 2fgA/M^4, \quad F^{14} = (g^2/M^2) A_u - A_x \\ F^{23} &= 2fA/M^4 \sin \alpha, \quad F^{24} = gA_y/M^2 \\ F^{34} &= gA_u/M^2 \sin \alpha. \end{aligned} \quad \dots(2.2)$$

Here and in what follows the following notations are used :

(i) A suffix indicate partial derivative

$$\text{e. g. } A_u = \frac{\partial A}{\partial u}, \quad A_{uy} = \frac{\partial^2 A}{\partial u \partial y} \text{ etc.,}$$

(ii) $2f$ stands for $g_\alpha + g \cot \alpha$;

(iii) The variable y is defined by the relation

$$dy = g d\alpha;$$

(iv) The semicolon indicates covariant derivative. Using (2.2) and the metric (1.8) it can be easily verified that the Maxwell equations $F^{ik};_k = 0$ for source free electromagnetic fields give rise to the following four differential equations for the function A :

$$\begin{aligned} A_{xx} + 2A_x (M_x/M) + (4f^2 A/M^4) &= 0 \\ A_{xy} + (2f A/M^2)_u &= 0, \\ A_{ux} - (2f A/M^2)_y &= 0, \\ (g^2/M^2) (A_{uu} + A_{yy}) + (2f A_y/M^2) \\ &- [A_{xu} + 2A_x (M_u/M)] = 0. \end{aligned} \quad \dots(2.3)$$

The electromagnetic energy tensor E_{lk} is given by

$$E_{lk} = -g^{lm} F_{ll} F_{km} + (\frac{1}{2}) g_{lk} F_{lm} F^{lm}.$$

Using F_{lk} , F^{ik} and g_{lk} given by (1.8) we have computed the components E_{lk} . The expressions for E_{lk} are slightly lengthy and therefore they are not given here. Using these expressions in the relations

$$E_{(ab)} = e^i_{(a)} e^k_{(b)} E_{ik}, \quad e^i_{(a)} \theta^a = dx^i, \quad \dots(2.4)$$

we can determine the tetrad components $E_{(ab)}$, of the electromagnetic energy tensor. They are given by

$$\begin{aligned} E_{(23)} &= E_{(24)} = E_{(34)} = E_{(44)} = 0 \\ E_{(11)} &= (g^2/M^2) \left(A_u^2 + A_y^2 \right) \\ E_{(12)} &= (g/M) (A_x A_y + 2 f A A_u/M^2) \\ E_{(13)} &= (g/M) (2 f A A_y/M^2 - A_u A_x) \\ E_{(14)} &= E_{(22)} = E_{(33)} = 1/2 (A_x^2 + 4 f^2 A^2/M^4). \end{aligned}$$

3. THE EINSTEIN-MAXWELL FIELD EQUATIONS

We shall use the following field equations :

$$\begin{aligned} R_{lk} - \frac{1}{2} g_{lk} R &= - 8 \pi E_{lk} - \Lambda g_{lk} \\ &- 8 \pi [(p + \rho) v_l v_k - p g_{lk} + \sigma w_l w_k] \end{aligned} \quad \dots(3.1)$$

with

$$g^{ik} v_l v_k = 1, \quad g^{ik} w_l w_k = 0, \quad g^{ik} v_l w_k = 1. \quad \dots(3.2)$$

The last in (3.2) is the normalising condition. Here $\sigma w_l w_k$ in the tensor arising out of the following null radiation, v^i is the flow vector of the perfect fluid and Λ is the cosmological constant. The other symbols occurring in (3.1) have their usual meanings. The field equations (3.1) can be expressed in the tetrad form as

$$\begin{aligned} R_{(ab)} &= \Lambda g_{(ab)} - 8\pi E_{(ab)} - 8\pi \sigma w_{(a)} w_{(b)} \\ &- 8\pi [(p + \rho) v_{(a)} v_{(b)} - \frac{1}{2} (\rho - p) g_{(ab)}]. \end{aligned} \quad \dots(3.3)$$

For the metric (1.8) and the tetrad (1.9) we take the tetrad components of v_l and w_l as

$$v_{(a)} = (1/2\lambda, 0, 0, \lambda), \quad w_{(a)} = (1/\lambda, 0, 0, 0) \quad \dots(3.4)$$

where λ is a function of co-ordinates to be determined from the field equations. It is painless to verify that v_l and w_l given by (3.4) satisfy the conditions (3.2). Using (3.4) in (3.3) we obtain

$$R_{(24)} = R_{(34)} = 0 \quad \dots(3.5)$$

$$R_{(12)} = - 8\pi E_{(12)}, \quad R_{(13)} = - 8\pi E_{(13)} \quad \dots(3.6)$$

$$8\pi p = \Lambda - R_{(14)} - 8\pi E_{(14)} \quad \dots(3.7)$$

$$8\pi (p + \rho) = - 2 [R_{(22)} + 8\pi E_{(22)} + R_{(14)} + 8\pi E_{(14)}] \quad \dots(3.8)$$

$$\lambda^2 = - R_{(44)}/8\pi (p + \rho) \quad \dots(3.9)$$

$$16\pi \sigma = - 4 \pi (p + \rho) + \frac{R_{(44)} [R_{(11)} + 8\pi E_{(11)}]}{4\pi (p + \rho)}. \quad \dots(3.10)$$

Where $E_{(ab)}$ are given by (2.5) and $R_{(ab)}$ are given by the expressions listed in the paper by Vaidya *et al.*⁵.

The next section will be devoted to the solution of the above Einstein-Maxwell field equations.

4. THE SOLUTIONS OF THE FIELD EQUATIONS

Equations (3.5) contain only the function M . One can easily verify that eqn. (3.5) admit the following solution,

$$M^2 = R^2 F^2 (X^2 + Y^2)/R_0^2 \quad \dots(4.1)$$

where

$$X = X(r), Y = Y(y), F = F(y), R = R(x) \quad \dots(4.2)$$

and

$$f = -F^2 Y Y_y. \quad \dots(4.3)$$

Here R_0 is a constant and the function X and Y satisfy the relations

$$\begin{aligned} X_{rr} &= -X/R_0^2, \quad X_r^2 = 1 - X^2/R_0^2 \\ Y_{yy} &= Y/R_0^2, \quad Y_y^2 = 1 + Y^2/R_0^2. \end{aligned} \quad \dots(4.4)$$

Using this M^2 and f in the system (2.3) of the four equations one can determine the function A as

$$A = \frac{eXX_r}{X^2 + Y^2} \quad \dots(4.5)$$

Where e is a constant of integration. Note the fact that A is a function of r and y only.

Using M^2 and A given by (4.5) in eqns. (3.6) a solution of these two differential equations can be expressed as

$$2L = \frac{R^2}{R_0^2} + \frac{mXX_r - 4\pi e^2 \left[1 - \left(2X^2/R_0^2 \right) \right]}{X^2 + Y^2} \quad \dots(4.6)$$

where m is a constant of integration.

If the function g is known, the function F can be determined from (4.3). This completes the task of determining all the metric potentials except the function $g(\alpha)$.

The pressure p , the density ρ , λ^2 and the radiation density σ can be determined from (3.7), (3.8), (3.9) and (3.10) respectively.

Since we are interested in the associated Kerr-Newman metric in the cosmological background, we replace the variable y by the variable ψ defined by

$$Y(y) = a [1 + \cos \psi \log \tan \psi/2] \quad \dots(4.7)$$

where a is an arbitrary constant. For our solution $g \sin \alpha$ must be

$$g \sin \alpha = a [\sin^2 \psi \log \tan \psi/2 - \cos \psi]. \quad \dots(4.8)$$

Using (4.7) and (4.8) eqn. (4.3) determines F as

$$F \sin \alpha = \sin \psi. \quad \dots(4.9)$$

The physical parameters p , ρ , λ^2 and σ are given by

$$8 \pi p = \Lambda + 2L \left(\frac{R_0^2}{R^4} - \frac{R_{xx}}{R} \right) - \frac{R^2}{R_0^2} \left[-\frac{R_{xx}}{R} - \frac{3R_x^2}{R^2} - \frac{2R_0^2}{R^4} \right] \quad \dots(4.10)$$

$$\begin{aligned} 8 \pi (p + \rho) = & - \frac{4R_0^2}{R^4} \left(L - \frac{R^2}{R_0^2} \right) - \frac{2RR_{xx}}{R_0^2} - \frac{2Y^2}{R_0^2 (X^2 + Y^2)} \\ & + \frac{R_x^2}{R^2} \left(4L - \frac{2R^2}{R_0^2} \right) \\ & + \frac{2R_x}{R^3 (X^2 + Y^2)} \left\{ mR_0^2 \left(1 - \frac{2X^2}{R_0^2} \right) + 16 \pi e^2 XX_r \right\} \end{aligned} \quad \dots(4.11)$$

$$\lambda^2 = 2 \left(\frac{R_0^2}{R^4} - \frac{R_{xx}}{R} \right) / 8 \pi (p + \rho) \quad \dots(4.12)$$

$$\begin{aligned} 16 \pi \sigma = & - 4 \pi (p + \rho) \\ & + \frac{2 \left(R_{xx} R^{-1} - R_0^2 R^{-4} \right)}{4 \pi (p + \rho)} \left[\frac{-R_x}{R (X^2 + Y^2)} \left\{ m \left(1 - \frac{2X^2}{R_0^2} \right) \right. \right. \\ & \left. \left. + \frac{16 \pi e^2 XX_r}{R_0^2} \right\} + \frac{2L^2 R_{xx}}{R} - \frac{2R_0^2}{R^4} \left(L - \frac{R_0^2}{R^2} \right) \right]. \end{aligned} \quad \dots(4.13)$$

The final form of the metric of our solution is given by

$$\begin{aligned} ds^2 = & 2 [du + a \{ \sin^2 \psi \log \tan \psi/2 - \cos \psi \} d\beta] dx \\ & - \frac{R^2}{R_0^2} (X^2 + Y^2) \left[\frac{d\psi^2}{1 + \frac{Y^2}{R_0^2}} + \sin^2 \psi d\beta^2 \right] \\ & - \left[\frac{R^2}{R_0^2} + \frac{m XX_r - 4\pi e^2}{X^2 + Y^2} \left(1 - \frac{2X^2}{R_0^2} \right) \right] [du + a \{ \sin^2 \psi \log \\ & \times \tan \psi/2 - \cos \psi \} d\beta]^2 \end{aligned} \quad \dots(4.14)$$

where $X = R_0 \sin (r/R_0)$ and Y is given by (4.7).

When $R = \text{constant} = R_0$ and $R_0 \rightarrow \infty$, the metric (4.14) reduces to

$$\begin{aligned} ds^2 = & 2 [du + a \{\sin^2 \psi \log \tan \psi/2 - \cos \psi\} d\beta] dx \\ & - (r^2 + Y^2) (d\psi^2 + \sin^2 \psi d\beta^2) \\ & - \left[1 + \frac{mr - 4\pi e^2}{r^2 + Y^2} \right] [du + a \{\sin^2 \psi \log \tan \psi/2 - \cos \psi\} d\beta]^2 \end{aligned} \quad \dots(4.15)$$

where Y is given by (4.7). The metric (4.15) is the associated Kerr-Newman metric discussed by Bhatt¹.

When $m = e = a = 0$, the metric (4.14) reduces to the Robertson-Walker metric (1.5).

Thus the metric (4.14) is such that it reduces to the associated Kerr-Newman metric in the vicinity of the source and it reduces to the Robertson-Walker metric in the absence of the source. Therefore (4.14) can be interpreted as associated Kerr-Newman metric in the cosmological background of closed Robertson-Walker universe.

A particular case, $R = \text{constant} = R_0$, of the metric (4.14) is note worthy. In this case the metric (4.14) takes in the form

$$\begin{aligned} ds^2 = & 2 [du + a \{\sin^2 \psi \log \tan \psi/2 - \cos \psi\} d\beta] dx \\ & - [X^2 + Y^2] \left[\frac{d\psi^2}{1 + \frac{Y^2}{R_0^2}} + \sin^2 \psi d\beta^2 \right] \\ & - \left[1 + \frac{mXX_r - 4\pi e^2 \left(1 - \frac{2X^2}{R_0^2} \right)}{X^2 + Y^2} \right] \\ & \times [du + a \{\sin^2 \psi \log \tan \psi/2 - \cos \psi\} d\beta]^2 \end{aligned} \quad \dots(4.16)$$

where $X = R_0 \sin (r/R_0)$ and Y is given by (4.7).

The physical parameters p , ρ , λ^2 and σ are given by

$$\begin{aligned} 8\pi p = & \Lambda + (2/R_0^2) (L - 1), \quad 8\pi (p + \rho) = -(4/R_0^2) (L - 1) - 2H \\ \lambda^2 = & \frac{2}{R_0^2 \cdot 8\pi (p + \rho)}, \quad 16\pi\sigma = \frac{H \left[\frac{4}{R_0^2} (L - 1) + H \right]}{\left[\frac{2}{R_0^2} (L - 1) + H \right]} \end{aligned} \quad \dots(4.17)$$

where

$$H = \frac{Y^2}{R_0^2 (X^2 + Y^2)}, \quad 2L = 1 + \frac{mXX_r + 4\pi e^2 \left(1 - \frac{2X^2}{R_0^2}\right)}{X^2 + Y^2}$$

$X = R_0 \sin(r/R_0)$ and Y is given by (4.7).

Now for a physically viable model the parameters p , ρ , λ^2 and σ should be positive. From the results (4.17) we have verified that the positivity of these four parameters imposes the following restrictions on Λ and r :

$$\Lambda \geq \frac{2}{R_0^2} (1 - L) \quad \dots(4.18)$$

and

$$X^2 \geq \left[mXX_r - 4\pi e^2 \left(1 - \frac{2X^2}{R_0^2}\right) \right] \quad \dots(4.19)$$

where L is given by (4.17).

When $R_0 \rightarrow \infty$, the inequality (4.19) becomes

$$r^2 \geq mr - 4\pi e^2. \quad \dots(4.20)$$

If we take the equality in (4.20), we get the familiar equation governing the horizons of the Reissner-Nordstrom solution.

When $m = e = a = 0$, the metric (4.16) reduces to the metric of Einstein static universe. In this case (4.18) gives $\Lambda \geq 1/R_0^2$. The metric (4.16) describes the associated Kerr-Newman metric in the background of Einstein's universe provided the restrictions (4.18) and (4.19) are satisfied.

When $a = 0$, the metric (4.14) represents the Reissner-Nordstrom metric in the background of closed Robertson-Walker universe. Here the radiation density σ is non-zero. But if, in addition, $R = \text{constant} = R_0$, σ vanishes and we get the Reissner-Nordstrom metric in the background of Einstein static universe.

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VIBRATION AND BUCKLING OF PARABOLICALLY TAPERED POLAR ORTHOTROPIC ANNULAR PLATES ON ELASTIC FOUNDATION

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The present paper analyses the effect of hydrostatic peripheral loading and that of elastic foundation on the natural frequencies of free axisymmetric vibrations and critical buckling parameter of parabolically tapered annular plates exhibiting polar orthotropy on the basis of classical plate theory. The governing fourth order linear differential equation has been solved by using Chebyshev polynomials for three different boundary conditions: inner and outer both clamped; inner clamped and outer simply supported; inner free and outer clamped. Transverse displacements, moments for a particular set of plate parameters and buckling loads in compression for different values of taper constant, foundation modulus for various values of rigidity ratios have been computed for first two modes of vibration. A comparison of our results with those available in literature shows a reasonable agreement.

NOMENCLATURE

r, θ	Polar co-ordinates of a point in the mid-plane of the plate
h	$h(r)$, thickness of the plate
h_0	non-dimensional thickness of the plate at the centre
a, b	outer and inner peripheral radii of the plate, respectively
ρ	mass density per unit volume
w	$w(r, t)$, displacement function
w, r	$\partial w / \partial r$
t	time
w, t	$\partial w / \partial t$
D_r	$E_r h^3 / 12 (1 - \nu_r \nu_\theta)$, flexural rigidity in radial direction
D_θ	$E_\theta h^3 / 12 (1 - \nu_r \nu_\theta)$, flexural rigidity in tangential direction
\bar{N}	$N / a D_0$, tensile in-plane force parameter
\bar{N}_{cr}	critical buckling load

Ω^2	$12\rho a^2\omega^2(1 - \nu_r\nu_\theta)/E_r h_0^2$ non-dimensional frequency parameter
ν_r	Poisson's ratio defined as strain in tangential direction due to unit strain in radial direction
ν_θ	Poisson's ratio defined as strain in radial direction due to unit strain in tangential direction
ω	circular frequency in radians per second
k_f	foundation modulus
K	$k_f a/E_r$
α	taper parameter
E_r, E_θ	Young's moduli in radial and tangential directions respectively
m	a positive integer, the number of collocation points
c_1, c_2, \dots, c_m	unknown constants in eqn. (5)
T_1, T_2, \dots, T_m	Chebyshev polynomials
y_0, y_1, \dots, y_{m-5}	zeros of the Chebyshev polynomial T_{m-4} in eqn. (7)
T_k^j	j th integral of T_k
C, S, F.	denote clamped, simply supported and free peripheries, respectively.

1. INTRODUCTION

In the recent past a considerable amount of work has been done on vibrations of annular plates of variable thickness due to their continually increasing use in the dynamic design of various engineering structures. A survey of literature by Leissa¹ shows that relatively little work has been done dealing with vibration of plates of variable thickness possessing polar orthotropy. There is no doubt that the consideration of the thickness variation together with anisotropy in structural components not only ensures reduction in size and weight maintaining high strength but also meets the desirability of economy. Keeping this in view fairly recent researches dealing with free transverse vibrations of circular/annular polar orthotropic plates have been reported²⁻⁶. Under normal working conditions, these plates may be subjected to in-plane stressing arising from hydrostatic, centrifugal and thermal⁷⁻⁹ and the supporting medium may be isotropic, homogeneous and linearly elastic : that is the Winkler type foundation.

In the work reported here the combined effects of elastic foundation and in-plane forces together with the polar orthotropy have been analysed on the axisymmetric vibrations of annular plates of parabolically varying thickness on the basis of classical theory of plates for the first two modes of vibration. The fourth order linear differen-

tial equation with variable coefficients which governs the motion of such plates, has been solved by Chebyshev polynomials.

2. ANALYSIS OF THE PROBLEM AND ITS APPROXIMATE SOLUTION

The differential equation which governs the axisymmetric vibrations of a homogeneous polar orthotropic annular plate of thickness $h(r)$ resting on an elastic foundation of modulus k_f and subjected to a tensile in-plane force N at the outer periphery is given by⁴

$$D_r w_{,rrrr} + [2(D_r + r D_{r,r})/r] w_{,rrr} + \{(-D_\theta + (2 + \nu_\theta) r D_{r,r} + r^2 (D_{r,rr} + N_r))/r^2\} w_{,rr} + \{D_\theta - r D_{\theta,r} + r^2 (\nu_\theta D_{r,rr} + N_r + r N_{r,r})/r^3\} w_{,r} + k_f w + \rho h w_{,tt} = 0 \quad \dots (1)$$

where

$N_r = - (N/h_a) (a^{\lambda+1}/(a^{2\lambda} - b^{2\lambda})) h \{r^{\lambda-1} - b^{2\lambda}/r^{\lambda+1}\}$, $h_a = h|_{r=a}$, a and b are the outer and inner radii respectively, w is the transverse deflection and other symbols have their usual meanings (see Nomenclature).

Introducing the non-dimensional variables $x = r/a$, $\bar{w} = w/a$, $\bar{h} = h/a$ and thickness variation in radial direction given by $\bar{h} = h_0 (1 - \alpha x^2)$, eqn. (1) becomes.

$$\sum_{i=0}^4 B_i \frac{d^i \bar{w}}{dx^i} = 0 \quad \dots (2)$$

where $\bar{w}(x, t) = W(x) e^{i\omega t}$ (for harmonic vibrations), $B_4 = (1 - \alpha x^2)^3$

$$B_3 = (2/x) [(1 - \alpha x^2)^2 (1 - 7\alpha x^2)]$$

$$B_2 = [(1 - \alpha x^2)^3 \{-p(1 - \alpha x^2) - 6\alpha x^2(2 + \nu_\theta)\} - 6\alpha x^2(1 - 6\alpha x^2 + 5\alpha^2 x^4)]/x^2 + N_x$$

$$B_1 = (1 - \alpha x^2) [p(1 - \alpha x^2)^2 (1 + 5\alpha x^2) - 6\nu_\theta \alpha x^2 (1 - 5\alpha x^2)]/x^3 + (N_x/x) + (N_{x,x})$$

$$B_0 = (k_f a/D_0) - (1 - \alpha x^2) \Omega^2, D_0 = E_r h_0^3 / 12 (1 - \nu_r \nu_\theta),$$

$$p = E_\theta/E_r, N_x = -\bar{N} \{(1 - \alpha x^2)/(1 - \alpha)(1 - \beta^2 \lambda)\} \{x^{\lambda-1} - \beta^{2\lambda}/x^{\lambda+1}\}$$

$$\lambda = \sqrt{p}, \epsilon = b/a, \bar{N} = N/aD_0 \text{ and } \Omega^2 = 12\rho a^2 \omega^2 (1 - \nu_r \nu_\theta)/E_r h_0^2.$$

Due to the presence of variable coefficients in eqn. (2), its closed form solution is not possible. Keeping this in view and approximate solution is obtained by applying the Chebyshev collocation technique, as follows.

By taking a new independent variable

$$y \equiv \{2x - (1 + \epsilon)\}/(1 - \epsilon), \quad \dots (3)$$

the range $\epsilon \leq x \leq 1$ is transformed to $-1 \leq y \leq 1$, the applicability range of the technique. Equation (2) now reduces to

$$\sum_{i=0}^4 A_i \frac{d^i W}{dy^i} = 0 \quad \dots(4)$$

where $A_i = \eta^i B_i$ ($i = 0, 1, 2, 3, 4$), and $\eta = 2/(1 - \epsilon)$. According to the present method, one has to proceed by assuming

$$\frac{d^4 W}{dy^4} = \sum_{i=0}^{m-5} c_{i+5} T_i \quad \dots(5)$$

where T_i ($i = 0, 1, 2, \dots, m-5$) are Chebyshev-polynomials. Successive integrations of eqn. (5) lead to

$$W = c_1 + c_2 T_1 + c_3 T_1^1 + c_4 T_1^2 + \sum_{i=0}^{m-5} c_{i+5} T_i^4 \quad \dots(6)$$

where c_j ($j = 1, 2, \dots, m$) are unknown constants and T_i^j represents the j th integral of T_i .

Substitution of W and its derivatives in eqn. (4) gives an equation in terms of the unknown constants c 's and Chebyshev polynomials T 's. The satisfaction of this resultant equation at $(m-4)$ collocation points given by

$$y_l = \cos \left(\frac{2l+1}{m-4} \frac{\pi}{2} \right), \quad l = 0, 1, 2, \dots, m-5 \quad \dots(7)$$

provides a set of $(m-4)$ equations in terms of the unknowns c_j ($j = 1, 2, \dots, m$), which can be written in the matrix form as

$$[M][Y] = [0] \quad \dots(8)$$

where M and Y are matrices of order $(m-4) \times m$ and $m \times 1$ respectively.

3. FREQUENCY EQUATIONS

The following three sets of boundary conditions have been considered in this work : (i) C-C, clamped at both the inner and outer edges; (ii) C-S, clamped at the inner and simply supported at the outer edge; (iii) F-C free at the inner and clamped at the outer edge. The relations which should be satisfied at a clamped, simply supported and free boundary are $W = dW/dy = 0$, $W = \eta (d^2 W/dy^2) + (v_0/x) (dW/dy) = 0$ and $\eta (d^2 W/dy^2) + (v_0/x) (dW/dy) = \eta \{ \eta (d^3 W/dy^3) + (1/x) (d^2 W/dy^2) \} - (p/x^2) (dW/dy) = 0$ respectively.

Applying the C-C boundary condition at $y = -1$ and $y = 1$ to the displacement function $W(y)$, one obtains a set of four homogeneous equations. These equations

together with the field equation (8) give a complete set of m equations in m unknowns, which can be denoted by

$$\left[\frac{M}{M_{CC}} \right] [Y] = 0 \quad \dots(9)$$

where M_{CC} is a matrix of order $4 \times m$.

For a non-trivial solution of eqn. (9), the frequency determinant must vanish and hence,

$$\left| \frac{M}{M_{CC}} \right| = 0. \quad \dots(10)$$

Similarly for C-S and F-C plates frequency determinant can be written as

$$\left| \frac{M}{M_{CS}} \right| = 0, \quad \left| \frac{M}{M_{FC}} \right| = 0 \quad \dots(11, 12)$$

respectively.

4. NUMERICAL RESULTS

In the work reported here, transcendental equations (10), (11) and (12) have been solved numerically for the first two modes of vibration. The effect of various plate parameters such as radii ratio b/a ($= 0.1, 0.3, 0.5$), foundation parameter K ($= 0.10, 0.02$), rigidity ratio E_θ/E_r ($= 0.5, 1.0, 2.0, 5.0$), in-plane force parameter \bar{N} ($= -20$ (10) 20) and taper constant α ($= -0.2$ 0.5) on the natural frequencies have been computed for three different boundary conditions C-C, CS, and F-C for $\nu_\theta = 0.3$. Nor-

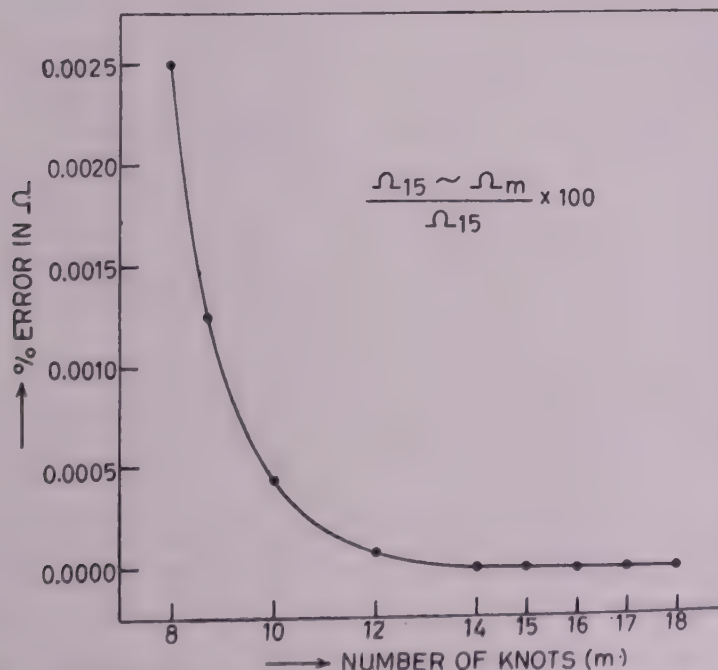


FIG. 1. % Error in Ω for C-C annular plate for $b/a = 0.5$, $E_\theta/E_r = 5.0$, $\bar{N} = 10.0$, $K = 0.02$ and $\alpha = -0.3$.

malized moments and displacements have been plotted for $b/a = 0.3$, $\alpha = \pm 0.3$, $\bar{N} = 10.0$, $E_\theta/E_r = 5.0$ and $K = 0.01$. Critical buckling loads have been computed for various values of the taper constant, foundation parameter, the rigidity and radii ratios for both the modes of vibration.

The number of collocation points have been taken as $m = 15$ since further increase in the value of m does not improve the results even in the fourth place of decimal. A consistent improvement was seen with the increase in the number of knots shown by Fig. 1 taking $b/a = 0.5$, $E_\theta/E_r = 5.0$, $\bar{N} = 10.0$, $K = 0.02$ and $\alpha = -0.3$ for C-C plate. All the computations were carried out in double precision arithmetic on DEC-2050 Digital Computer.

5. DISCUSSIONS

The results are presented in Figures (2-9) and Tables (1-4). It is found that the frequency parameter for a C-S plate is greater than that for a F-C plate but less than that for C-C plate for the same set of plate parameters. Fig. 2 showing the plots for Ω versus E_θ/E_r for $b/a = 0.3$, $\bar{N} = 10.0$, $K = 0.01$ and $\alpha = \pm 0.1$ reveals that the frequency parameter Ω increases with the increasing values of E_θ/E_r for all the three boundary conditions for plates vibrating in fundamental mode. A similar inference can be drawn from Fig. 3 showing the plots for second mode.

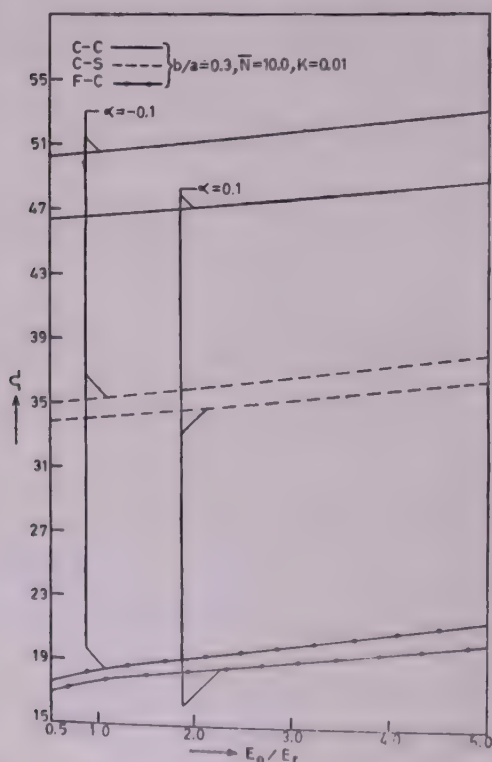


FIG. 2. Natural frequency for C-C, C-S and C-F annular plates vibrating in fundamental mode.

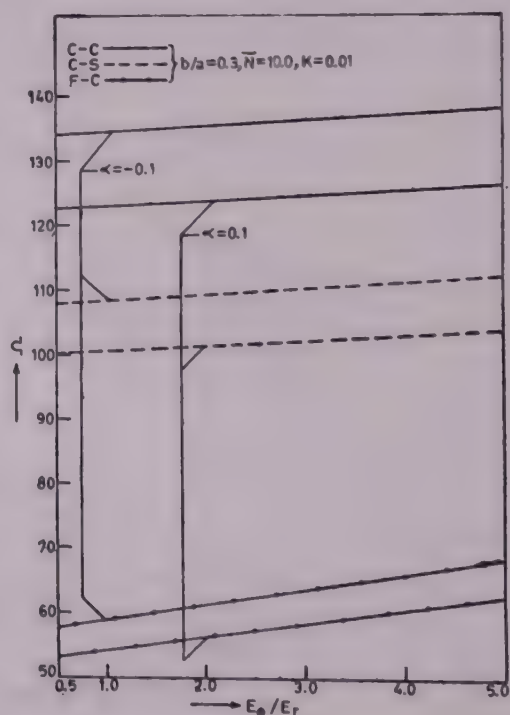


FIG. 3. Natural frequency for C-C, C-S and F-C annular plates vibrating in second mode.

Figure 4 shows the effect of thickness variation on the natural frequencies for $b/a (= 0.3, 0.5)$, $E_\theta/E_r (= 1.0, 5.0)$, $K = 0.01$ and $\bar{N} = 10.0$ for the fundamental mode. It is observed that frequency parameter Ω increases as the outer periphery becomes thicker and thicker (i. e. for decreasing values of α) for all the three plates. However the rate of increase of Ω for C-S plate is less than that for C-C plate but greater than that of F-C plate, other plate parameters being fixed. As far as the behaviour in the second mode is concerned, plots have been given in Fig. 5 which show that the rate of increase of Ω with decreasing α is higher as compared to the fundamental mode.

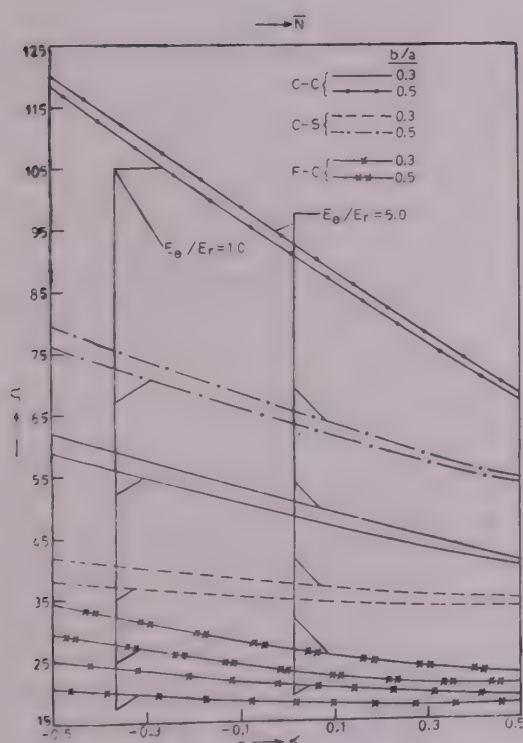


FIG. 4. Natural frequencies of annular plates vibrating in fundamental mode for $K=0.01$ and $\bar{N} = 10.0$.

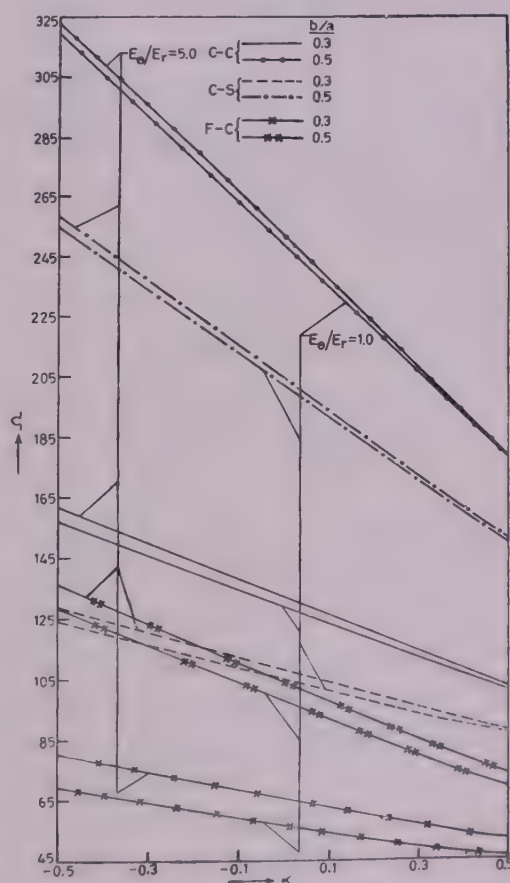


FIG. 5. Natural frequencies of annular plates vibrating in second mode for $K = 0.01$ and $N = 10.0$.

Figure 6 shows the effect of in-plane force parameter on the natural frequencies in the presence of elastic foundation $K = 0.02$. It is observed that the frequency parameter Ω increases as the in-plane force parameter \bar{N} increases for all the boundary conditions considered here. The effect of orthotropy is found to decrease with the increasing values of \bar{N} for all the three boundary conditions. A similar behaviour of frequency parameter for the second mode is observed from Fig. 7.

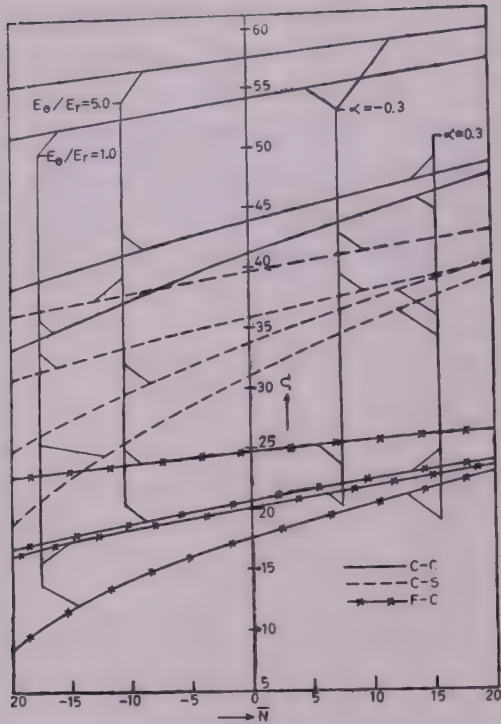


FIG. 6. Natural frequencies of annular plates for first mode of vibrations for $b/a = 0.3$ and $K = 0.02$.

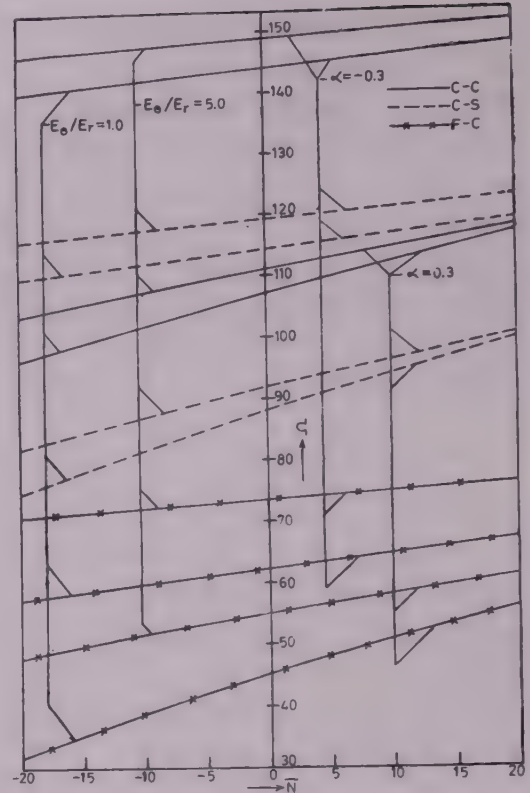


FIG. 7. Natural frequencies of annular plates for second mode of variations for $b/a = 0.3$ and $K = 0.02$.

In Figures 8 (a, b, c) normalized displacements and moments are shown for $\bar{N} = 10.0$, $K = 0.01$, $b/a = 0.3$, $E_\theta/E_r = 5.0$ and $\alpha = \pm 0.3$ for the first two modes of vibrations. It is clear from the plots that the lines along which moments vanish shift towards the outer edge as the plate becomes thinner and thinner at the outer edge for all the three boundary conditions except for F-C plate vibrating in second mode (where the behaviour is just the reverse). The radii of nodal circles for $\alpha = -0.3$ are less than those for $\alpha = 0.3$ for all the three cases. It is seen that the transverse deflection for a F-C plate for $\alpha = -0.3$ is always less than the corresponding deflection for $\alpha = 0.3$, but this is not the case with C-C and C-S plates. For these cases the transverse deflection for $\alpha = -0.3$ is greater than those for $\alpha = 0.3$ towards the inner edge and vice versa towards the outer edge.

The critical values \bar{N}_{cr} of \bar{N} , corresponding to the critical buckling load in compression, for $K (= 0.00, 0.01, 0.02)$, $b/a (= 0.1, 0.3, 0.5)$, $\alpha = -0.5 (0.2) 0.5$ and $E_\theta/E_r = 5.0$ for all the three plates are given in Tables (1–3) for both the modes of vibration. Figure 9 shows the plots for Ω versus \bar{N} for $b/a = 0.3$, $K = 0.01$, $E_\theta/E_r = 1.0, 5.0$ and $\alpha = \pm 0.3$ for all the three plates vibrating in the fundamental mode. It is evident that the buckling loads for C-C plate is higher than those for C-S and F-C plates, whatever may be the value of the taper constant α , the radii ratio b/a and the foundation parameter K .

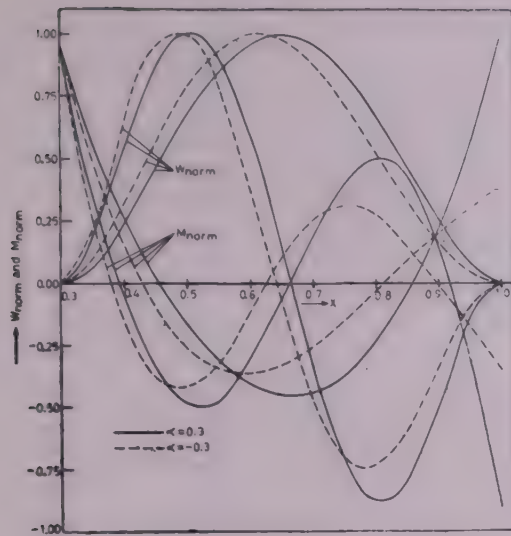


FIG. 8a. Normalized displacements and moments F-C annular plate for first two modes of vibrations for $b/a = 0.3$, $\bar{N} = 10.0$, $K = 0.01$ and $E_\theta/E_r = 5.0$.

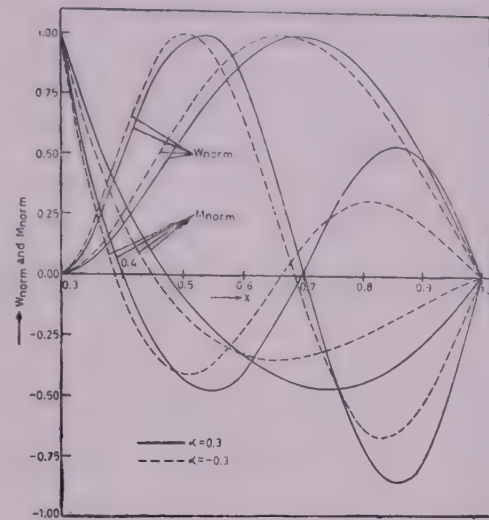


FIG. 8b. Normalized displacements and moments for C-S annular plate for first two modes of vibrations for $b/a = 0.3$, $\bar{N} = 10.0$, $K = 0.01$ and $E_\theta/E_r = 5.0$.

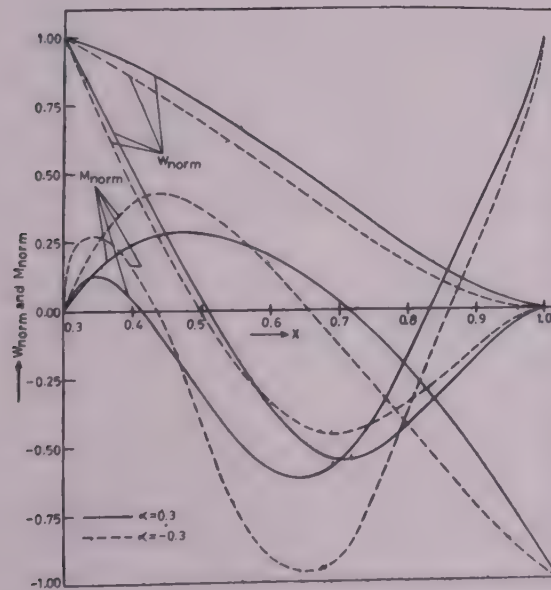


FIG. 8c. Normalized displacements and moments for C-C annular plate for first two modes of vibrations for $b/a = 0.3$, $\bar{N} = 10.0$, $K = 0.01$ and $E_\theta/E_r = 5.0$.

It has been found that the presence of elastic foundation increases the value of critical buckling load. Further, the increase in orthotropy also increases the values of \bar{N}_{cr} in all the cases.

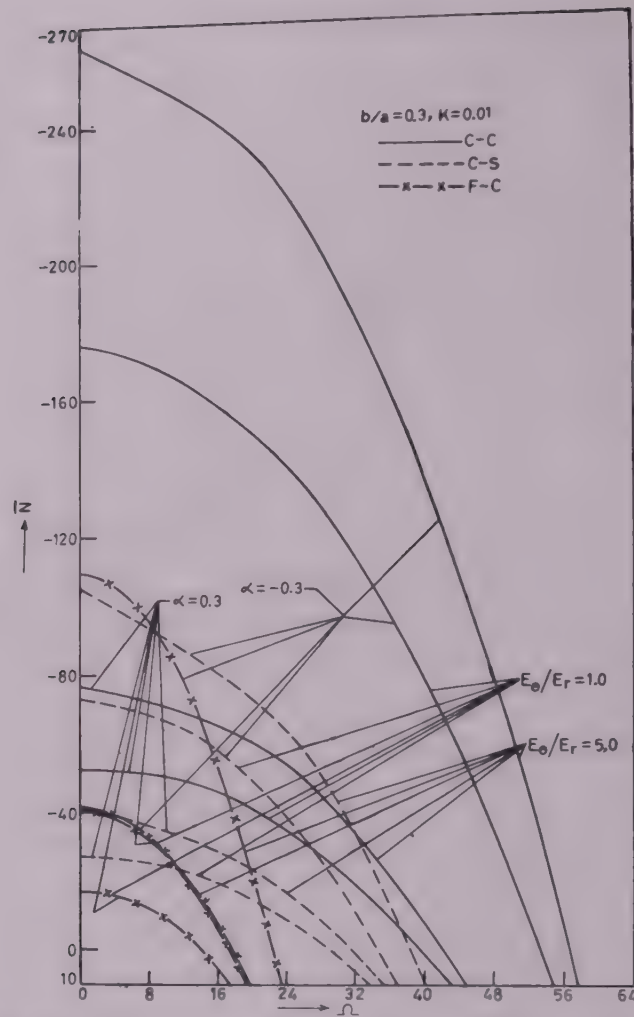


FIG. 9. Buckling loads for C-C, F-C and F-C annular plates for fundamental mode.

Table 4 shows comparison of the results with those available in the literature. In this Table, our results for isotropic/orthotropic annular plates of uniform thickness show very good agreement with those of^{10,11}.

6. CONCLUSIONS

The effect of polar orthotropy together with elastic foundation and in-plane forces on axisymmetric vibrations of annular plates of parabolically varying thickness has been analysed on the basis of classical plate theory. The vibrational characteristics of annular plates in absence of peripheral loading (i. e. $\bar{N} = 0.0$) as well as elastic foundation ($K = 0$) have been studied as special cases.

It has been found that Ω and critical buckling load \bar{N}_{cr} increase for the increasing values of foundation parameter, rigidity and radii ratios and in-plane force parameter for all the three boundary conditions whatever may be the value of taper constant. Thus the desired frequency can be achieved for a specified plate by a proper choice of various

TABLE 1

Critical buckling loads in compression for C-C plate for $E_\theta/E_r = 5.0$, $\nu_\theta = 0.3$.

$b/a \downarrow \vec{\alpha}$	mode \downarrow	-0.5	-0.3	-0.1	0.0	0.1	0.3	0.5
$K = 0.0$								
0.1	I	-253.62	-188.95	-135.20	-112.14	-91.47	-56.94	-30.89
	II	-506.35	-378.07	-270.34	-223.85	-182.10	-112.35	-60.04
0.3	I	-349.89	-255.91	-179.18	-146.79	-118.14	-71.31	-37.26
	II	-711.87	-521.35	-364.93	-298.70	-243.04	-144.10	-74.55
0.5	I	-677.67	-481.53	-325.88	-261.87	-130.64	-81.83	-45.46
	II	-1399.86	-994.58	-672.56	-540.04	-206.35	-147.59	-77.35
$K = 0.01$								
0.1	I	-268.33	-202.22	-116.01	-123.02	-78.45	-65.11	-37.00
	II	-511.76	-382.95	-228.71	-227.87	-153.12	-115.43	-62.43
0.3	I	-358.45	-263.80	-186.32	-153.52	-124.43	-76.63	-41.43
	II	-714.62	-523.89	-367.23	-300.88	-242.07	-145.83	-75.93
0.5	I	-681.64	-485.37	-329.44	-265.26	-209.60	-121.38	-60.56
	II	-1401.12	-995.77	-673.65	-541.09	-426.07	-244.28	-119.54
$K = 0.2$								
0.1	I	-282.76	-167.22	-120.39	-133.52	-82.38	-72.79	-42.52
	II	-517.22	-323.39	-229.92	-231.97	-154.21	-118.61	-64.94
0.3	I	-366.96	-271.64	-193.39	-160.18	-130.64	-81.83	-45.46
	II	-717.38	-526.44	-369.55	-303.06	-244.12	-147.59	-77.35
0.5	I	-685.82	-489.19	-332.98	-268.64	-212.76	-124.17	-62.83
	II	-1402.38	-996.95	-674.75	-542.14	-427.07	-245.25	-120.26

TABLE 2

Critical buckling loads in compression for C-S plate for $E_\theta/E_r = 5.0$, $\nu_\theta = 0.3$.

$h/a \downarrow \vec{\alpha}$	mode \downarrow	-0.5	-0.3	-0.1	0.0	0.1	0.3	0.5
$K = 0.0$								
0.1	I	-90.81	-70.14	-52.59	-44.85	-37.74	-25.27	-15.90
	II	-360.15	-270.21	-194.46	-161.67	-132.13	-82.47	-44.79
0.3	I	-130.08	-98.29	-71.67	-60.14	-49.71	-31.96	-18.11
	II	-501.29	-371.68	-261.58	-214.82	-173.29	-105.05	-55.10
0.5	I	-259.25	-189.09	-132.15	-108.22	-87.11	-52.72	-27.76
	II	-997.27	-708.79	-481.95	-387.28	-306.29	-176.73	-87.23
$K = 0.01$								
0.1	I	-102.22	-80.53	-61.85	-53.50	-45.74	-31.84	-19.95
	II	-368.82	-227.97	-201.27	-167.98	-137.94	-87.22	-48.39
0.3	I	-137.29	-104.97	-77.75	-65.88	-55.09	-36.53	-21.70
	II	-509.85	-375.79	-265.27	-218.29	-176.53	-107.77	-57.25
0.5	I	-262.83	-192.47	-135.29	-111.23	-89.96	-55.21	-29.81
	II	-997.63	-711.60	-483.69	-389.64	-307.87	-178.08	-88.34
$K = 0.02$								
0.1	I	-113.20	-90.35	-70.59	-61.59	-53.14	-37.74	-24.13
	II	-337.64	-285.89	-208.25	-174.48	-143.94	-92.17	-52.19
0.3	I	-144.42	-111.55	-83.70	-71.48	-60.31	-40.90	-25.04
	II	-51.439	-379.94	-269.03	-221.83	-179.14	-110.59	-59.50
0.5	I	-266.40	-195.93	-138.41	-114.21	-92.79	-57.68	-31.82
	II	-999.65	-713.50	-485.44	-391.31	-309.43	-179.45	-89.46

TABLE 3

Critical buckling loads in compression for F-C plate for $E_\theta/E_r = 5.0$, $\nu_\theta = 0.3$.

$b/a \downarrow$	\vec{a} mode \downarrow	-0.5	-0.3	-0.1	0.0	0.1	0.3	0.5
$K = 0.0$								
0.1	I	-115.38	-82.99	-57.36	-46.80	-37.61	-22.96	-37.39
	II	-320.73	-242.10	-176.17	-147.57	-121.65	-76.73	-59.68
0.3	I	-118.95	-85.31	-58.87	-48.03	-38.61	-23.58	-39.49
	II	-352.15	-257.96	-181.04	-148.63	-120.01	-73.38	-72.62
0.5	I	-141.78	-97.79	-64.47	-51.28	-40.12	-23.11	-46.05
	II	-564.43	-397.73	-266.49	-212.49	-166.77	-94.62	-124.23
$K = 0.01$								
0.1	I	-155.48	-118.94	-89.13	-76.43	-65.05	-45.71	-30.05
	II	-350.43	-261.99	-187.89	-155.95	-127.27	-79.34	-43.29
0.3	I	-145.14	-109.70	-81.31	-61.40	-58.84	-49.22	-27.35
	II	-358.53	-264.38	-187.03	-154.20	-125.03	-76.95	-41.43
0.5	I	-155.58	-110.92	-76.86	-63.24	-51.62	-33.50	-20.86
	II	-566.71	-400.37	-269.37	-215.85	-169.64	-97.17	-47.92
$K = 0.02$								
0.1	I	-189.498	-149.92	-116.74	-102.20	-88.82	-64.80	-42.12
	II	-359.715	-270.37	-195.33	-162.93	-133.81	-99.53	-49.59
0.3	I	-170.57	-133.52	-103.25	-90.30	-78.60	-58.32	-40.86
	II	-364.13	-269.53	-191.68	-158.58	-129.12	-80.42	-44.30
0.5	I	-169.24	-123.67	-89.16	-75.12	-63.01	-43.74	-29.51
	II	-569.85	-403.36	-272.18	-218.56	-172.68	-99.53	-50.04

TABLE 4

Comparison of frequencies for Isotropic/Orthotropic annular plates of uniform thickness
 $\bar{N} = 0.0$, $K = 0.0$, $\nu_\theta = 0.3$, $b/a = 0.3$, $\alpha = 0$.

Boundary condition \downarrow	mode \downarrow	Values of Rigidity Ratio E_θ/E_r	
		1.0	5.0
C-C	I	45.35	48.35
		45.29*	48.32**
		45.28**	
	II	125.35	
		125.39*	129.59
		29.98	33.27
C-S	I	29.90*	32.99**
		29.93**	
	II	100.42	104.77
		100.44*	
		11.42	
F-C	I	11.41*	16.33
	II	51.74	52.50
		51.71*	

*Values taken from Vogel and Skinner.¹⁰**Values from Gorman¹¹.

plate parameters. The above analysis will be of great help to design engineers for increasing the buckling load of structural components.

A close agreement of our results with those of exact values of Vogel and Skinner¹⁰ and that of Gorman¹¹ by finite element technique shows the versatility of the present approach.

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